The South African Mathematical Olympiad
Third Round 2011
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

Solutions

1. Since $45^2 = 2025$ and $46^2 = 2116$, there are precisely 45 perfect squares ≤ 2056 which are left out from the sequence of positive integers. Since $2056 - 45 = 2011$, we conclude that the 2011^{th} element is 2056.

2. Note that

$$\tfrac{5}{3} = 4^x - 4^y = (2^x - 2^y)(2^x + 2^y) = 1 \cdot (2^x + 2^y).$$

Therefore,

$$2^x = \frac{(2^x + 2^y) + (2^x - 2^y)}{2} = \frac{\frac{5}{3} + 1}{2} = \frac{4}{3}$$

and

$$2^y = \frac{(2^x + 2^y) - (2^x - 2^y)}{2} = \frac{\frac{5}{3} - 1}{2} = \frac{1}{3},$$

which implies

$$2^{x-y} = \frac{2^x}{2^y} = \frac{4/3}{1/3} = 4$$

and thus $x - y = 2$.

3. Let the sequence be $x+1$, $x+2$, $\ldots$, $x+m$. For such a sequence to be friendly, k must divide $x + k$ for all $1 \leq k \leq m$, which implies that $x = (x + k) - k$ must be divisible by all such k.

In the case $m = 20$, we can conclude that, in particular, x is divisible by 16. Therefore $x + 20$ is not divisible by 16 and thus also not by $20^2 = 16 \cdot 25$. Hence there is no friendly sequence in this case.

If $m = 11$, then x must be divisible by all numbers between 1 and 11 and thus also by their lcm, which is 27720. Hence we are looking for an $x = 27720y$ such that $x + 11 = 27720y + 11 = 11(2520y+1)$ is divisible by 11^2. This holds if $2520y+1 = (11\cdot229+1)y+1 = 11\cdot229y+y+1$ is divisible by 11. One possible solution is therefore $y = 10$, which yields $x = 277200$ and thus the friendly sequence

$$277201, 277202, \ldots, 277211.$$

Remark: One can avoid the calculation by noticing that $11! + 1$, $11! + 2$, $\ldots$, $11! + 11$ is a friendly sequence, making use of Wilson's theorem.

4. We prove by induction that there are 2^{n-1} possible networks fulfilling this condition if there are n airports. For $n = 2$, this is obvious (there can be a connection between the two airports or not).

   For the induction step, let H and L be the airports whose priorities are highest and lowest, respectively, and consider the following two possibilities:

Case 1: H and L are connected. Then there are direct connections from H to all other airports as well. Now the condition is always trivially satisfied if H is involved, and we only have to consider the remaining $n-1$ airports. There are 2^{n-2} possible networks between these airports by the induction hypothesis.

Case 2: H and L are not connected. Then L cannot be connected to any of the airports, and we can ignore L. By the induction hypothesis, there are 2^{n-2} feasible networks that connect the remaining $n-1$ airports.

   Altogether, we have $2^{n-2} + 2^{n-2} = 2^{n-1}$ possible networks, which completes the induction.

5. It is well known that property (b) is satisfied by any polynomial $f(x)$. In particular, it is satisfied for $f(x) = 0$, $f(x) = x$, $f(x) = x^2$ and $f(x) = x(x-1)$. Furthermore, these polynomials all attain nonnegative integer values if $x \in \mathbb{N}_0$, and they all satisfy property (a). We now show that these are the only four solutions.

   By property (a), we have $0 \le f(0) \le 0$ and thus $f(0) = 0$. By the same argument, $f(1) \in \{0, 1\}$ and $f(2) \in \{0, 1, 2, 3, 4\}$. Since $f(2) - f(0)$ has to be divisible by 2 (property (b)), we have $f(2) \in \{0, 2, 4\}$.

   Note also that $\gcd(n, n-1) = \gcd(n-1, n-2) = 1$ and $\gcd(n, n-2) \in \{1, 2\}$, which implies that

$$\operatorname{lcm}(n, n-1, n-2) = \begin{cases} n(n-1)(n-2) & n \text{ odd}, \\ \frac{n(n-1)(n-2)}{2} & n \text{ even}, \end{cases}$$

   and thus

$$\operatorname{lcm}(n, n-1, n-2) \ge \frac{n(n-1)(n-2)}{2} = n \cdot \frac{n^2 - 3n + 2}{2} > n^2 \cdot \frac{n-3}{2} \ge n^2 \qquad (1)$$

   for $n \ge 5$.

   We distinguish six cases:

   (a) $f(0) = f(1) = f(2) = 0$: In this case, $f(n)$ has to be divisible by n, $n-1$ and $n-2$ for all n and therefore also by $\operatorname{lcm}(n, n-1, n-2)$. In view of (1) and property (a), the only possibility is $f(n) = 0$ for all $n \ge 5$. In particular, $f(100) = 0$, so $f(3)$ and $f(4)$ have to be divisible by 97 and 96 respectively, implying $f(3) = f(4) = 0$ as well (since $0 \le f(3), f(4) \le 16$). Hence $f(n) = 0$ for all n.

   (b) $f(0) = 0$, $f(1) = 1$, $f(2) = 2$: In this case, $f(n)$, $f(n) - 1$ and $f(n) - 2$ have to be divisible by n, $n-1$, $n-2$ respectively, and so $f(n) - n$ has to be divisible by n, $n-1$ and $n-2$ for all n. As in the first case, we conclude that $f(n) - n$ has to be 0 (and thus $f(n) = n$) for all $n \ge 5$. In particular, $f(100) = 100$, and condition (b) again shows that $f(3) = 3$ and $f(4) = 4$ are the only possibilities, so $f(n) = n$ for all n.

(c) $f(0) = 0$, $f(1) = 0$, $f(2) = 2$: In this case, $f(n) - n(n-1)$ has to be divisible by n, $n-1$ and $n-2$ for all n, and we argue as before to obtain $f(n) = n(n-1)$.

(d) $f(0) = 0$, $f(1) = 1$, $f(2) = 4$: In this case, $f(n) - n^2$ has to be divisible by n, $n-1$ and $n-2$ for all n, and we argue as before to obtain $f(n) = n^2$.

(e) $f(0) = 0$, $f(1) = 1$, $f(2) = 0$: Now $f(5)$ has to be divisible by both 3 and 5, and thus $f(5) \in \{0, 15\}$. But in either case, $f(5) - f(1)$ is not divisible by 4.

(f) $f(0) = 0$, $f(1) = 0$, $f(2) = 4$: Now $f(5)$ has to be divisible by both 4 and 5, and thus $f(5) \in \{0, 20\}$. But in either case, $f(5) - f(2)$ is not divisible by 3.

We conclude that $f(x) = 0$, $f(x) = x$, $f(x) = x^2$ and $f(x) = x(x-1)$ are the only four possibilities.

6. Note first that $AE = AF$, $BF = BD$ and $CD = CE$, so that AD, BE and CF are concurrent by Ceva's theorem. Hence J lies on the line AD. Moreover, AG is an angle bisector in the isosceles triangle AEF and thus perpendicular to EF. Since JK is also perpendicular to EF, AG and JK must be parallel. So we can deduce that triangles ADG and JDK are similar, so that

$$\frac{GK}{DK} = \frac{AJ}{DJ}. \tag{2}$$

Now we consider the line BE, which intersects the sides of triangle ACD in B, J and E. Menelaus' theorem yields

$$\frac{AJ \cdot BD \cdot CE}{DJ \cdot BC \cdot AE} = 1$$

or

$$\frac{AE}{CE} = \frac{AJ \cdot BD}{DJ \cdot BC}.$$

Likewise, we apply Menelaus' theorem to triangle ABD, whose sides are intersected by CF in C, J and F, to obtain

$$\frac{AF}{BF} = \frac{AJ \cdot CD}{DJ \cdot BC}.$$

Adding the two, we find

$$\frac{AE}{CE} + \frac{AF}{BF} = \frac{AJ}{DJ \cdot BC} \cdot (BD + CD) = \frac{AJ}{DJ \cdot BC} \cdot BC = \frac{AJ}{DJ}. \tag{3}$$

Combining (2) and (3), we end up with the desired identity.