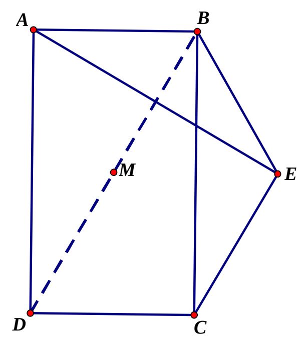
The South African Mathematical Olympiad
Third Round 2009
Senior Division (Grades 10 to 12)
Solutions

1. Note that $2009 = 7^2 \cdot 41$. Therefore, 41 must divide $n^2(n-1)$, which implies that 41 is a factor of either n or $n-1$. In particular, $n \geq 41$. For $n = 41$, neither n nor $n-1$ is divisible by 7, so this is not a solution. For $n = 42$, however, $n-1 = 41$, and n^2 is divisible by 7^2 since n is divisible by 7. Therefore, $n = 42$ has the desired property, and it is the smallest possible solution.

2.



Solution 1: Let M be the centre of the rectangle and F the reflection of E with respect to M. We show that $ABECDF$ is a regular hexagon, from which it follows immediately that the ratio is $\sqrt{3}$.

By the conditions of the problem, we have $AB = BE$ (reflection) and $BE = EC$ (given), so that

$$AB = BE = EC = CD = DF = FA$$

by symmetry. Now note that $\angle BAD = \angle BED = 90^o$ (by the construction of E), which implies that A, B, E, D lie on a circle with diameter BD, i.e. the circumcircle of $ABCD$. By symmetry, F lies on the circumcircle as well. Hence, $ABECDF$ must be a regular hexagon.

Solution 2: Note that $\angle AEB = \angle EAB$ (definition of E) $= \angle ADB$ (similar triangles). Hence, $ABED$ is a cyclic quadrilateral, and we can argue as in the previous solution.

Solution 3: Since $BM = BD$ is the perpendicular bisector of AE, we have

$$ME = MA = MB = MC = MD,$$

and we can conclude again that A, B, E, C, D, F lie on a circle. Now continue as in Solution 1.

Solution 4: The quadrilateral $ABEM$ is a kite (since the diagonals are perpendicular and $AB = BE$), and since AB and ME are parallel, it is a rhombus. Since also $AM = BM$, ABM is an equilateral triangle (and so is BEM). It follows immediately that the ratio of the sides of the rectangle is $\sqrt{3}$.

Solution 5: Coordinate geometry leads to the correct result as well. Take $A = (0, 0)$, $B = (a, 0)$, $C = (a, b)$ and $D = (0, b)$. Then, the coordinates of E are found to be $\left(\frac{2ab^2}{a^2+b^2}, \frac{2a^2b}{a^2+b^2}\right)$. The condition $EB = EC$ yields the equation

$$\frac{2a^2b}{a^2+b^2} = \frac{b}{2},$$

which can be reduced to $3a^2 = b^2$ or $\frac{b}{a} = \sqrt{3}$.

3. **Solution 1:** The sum of the numbers is $1 + 2 + \cdots + 10 = 55$; each number contributes to three of the new numbers, so the total of the new numbers is $3 \cdot 55 = 165$. Assume that none of the new numbers is greater than 17. If more than five of the new numbers are equal to 17, then there are two neighbours with the same new numbers; suppose they sit on seats x and $x + 1$. This implies that the girls in position $x - 1$ and $x + 2$ (possibly taken modulo 10) had the same old number, which is impossible.

   Therefore, there are not more than five girls whose new number is 17, and the sum of all the new numbers is at most $5 \cdot 16 + 5 \cdot 17 = 5 \cdot 33 = 165$. Since equality holds, there are precisely five girls whose new number is 16, and five girls whose new number is 17. By the above argument, the numbers of girls who sit next to each other must be distinct, and so they must form a 16–17–16... pattern. Consider two girls on seats x and $x + 1$ again. Their new numbers are 16 and 17, which shows that the old numbers of the girls in position $x - 1$ and $x + 2$ differ by exactly 1 (and this holds for any x!). If y is the position of the girl whose old number was 1, then this argument shows that the girls in positions $y - 3$ and $y + 3$ are either 0 (which is impossible) or 2. But there was only one girl who got number 2, and so we finally arrive at a contradiction.

   Solution 2: (by Desi Nikolov) Divide the nine girls with numbers 2 to 10 into three groups of three: the three girls to the left of no 1, the three girls to the right of no 1, and the three girls opposite no 1. The total sum of their numbers is 54, so the sum of at least one group must be at least 18. Therefore the new number of the girl in the middle of that group is at least 18.

4. Put subscripts on P, S, T to indicate their dependence on n. Then

$$T_1 - S_1 - 2 + 2P_1 = 1/x_1 - 2 + x_1 = (1/x_1 - 1)(1 - x_1) > 0.$$

   This gives the base case for an induction proof. For the induction step,

$$\begin{aligned} T_n - S_n - 2 + 2P_n &= (T_{n-1} - S_{n-1} - 2 + 2P_{n-1}) + \frac{1}{x_n} - x_n - 2(1 - x_n)P_{n-1} \\ &> \frac{1}{x_n} - x_n - 2(1 - x_n) \\ &= (1/x_n - 1)(1 - x_n) > 0. \end{aligned}$$

5. Observe that the sum of the positions of all the pebbles remains constant (and thus equal to 2009) throughout the process, since one of the positions increases by 1 while another decreases by 1 at each step. Therefore, no pebble can ever be placed in a hole with a label greater than 2009, implying that there is only a finite number of possible positions. Furthermore, no position

can be repeated: assume that the same position occurs twice, and let ℓ be the largest position of a hole from which pebbles have been removed between the two occurrences of the position. Then, the number of pebbles in hole $\ell + 1$ must have increased, an obvious contradiction.

Hence, the process terminates. At the end, all holes except for hole 0 are either empty or contain exactly one pebble. Let k be the largest number of a nonempty hole at the end of the process, and let $\ell < k$ be the smallest position of an empty hole (if any). At some step, pebbles must have been removed from hole ℓ (otherwise, there cannot be any pebbles in higher positions). Suppose step x is the last of these steps; no pebble may have been removed from hole $\ell + 1$ after step x (since hole ℓ is empty at the end). Thus, this hole was empty before step x. Now assume that step y was the last step at which pebbles were removed from hole $\ell+1$, and repeat the argument. This shows that holes $\ell+1$, $\ell+2$, ..., k all contain a pebble at the end of the process.

Therefore, there are only two possibilities:

- Holes 1, 2, ..., k all contain exactly one pebble, hole 0 contains $2009 - k$ pebbles. The sum of all the positions is thus $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$.
- Holes 1, 2, ..., k all contain exactly one pebble, except for the empty hole ℓ, and hole 0 contains $2010 - k$ pebbles. The sum of all the positions is $\frac{k(k+1)}{2} - \ell$.

Hence, we have $2009 = \frac{k(k+1)}{2} - \ell$, where $0 \leq \ell < k$. Since $\frac{k(k+1)}{2} - k = \frac{k(k-1)}{2}$, this representation is unique. We have

$$2009 = \frac{63 \cdot 64}{2} - 7,$$

which implies $k = 63$ and $\ell = 7$. Finally, we find that hole 0 contains $2010 - 63 = 1947$ pebbles.

6. Since $f(0) = \frac{1}{2}f(0)$, we deduce that $f(0) = 0$ and $f(\frac{1}{2}) = 1$. In the general case, it is useful to think in terms of the binary representation $B_2(x)$ of x.

For $\frac{1}{2} < x < 1$, $B_2(x)$ contains at least two $1's$, and we can express x as $x = \frac{1}{2} + 2^{-m} + 2^{-m}y$, where $m > 1$ and $0 \leqslant y < 1$. Then

$$f(x) = 1 - \tfrac{1}{2}f(2^{-m+1} + 2^{-m+1}y) = 1 - 2^{-m+1}f(\tfrac{1}{2} + \tfrac{1}{2}y) = 1 - 2^{-m+1} + 2^{-m}f(y).$$

In particular, when $y = 0$, we have $f(x) = 1 - 2^{-m}$. Thus

$$f(\tfrac{1}{2} + 2^{-m} + 2^{-m}y) = f(\tfrac{1}{2} + 2^{-m}) + f(2^{-m}y).$$

By repeating the argument, we obtain: if $x = z+2^{-m}y$, where $B_2(z)$ has length m and contains an even number of ones, and $0 \leqslant y < 1$, then $f(x) = f(z) + 2^{-m}f(y) = f(z) + f(2^{-m}y)$.

Assuming for the moment that

$$0 \leqslant f(y) < 2 \;\text{ for rational } x \tag{1}$$

(which we will prove later), we obtain the following procedure for calculating $B_2(f(x))$ from $B_2(x)$:

(a) Partition $B_2(x)$ into chunks of the form `00...0` and `100...01` (the latter case includes `11`), and possibly a final chunk of just a single `1`.

(b) Leave the chunks of zeros unchanged and replace the other chunks by `111...10` (this includes `11` $\to$ `10`).

(c) If there is a final `1`, change the last `0` to `1`.

Let the numbers represented by the chunks and their replacements respectively be u_j and $f(u_j)$. Then the procedure implements the identity

$$f(u_1 + u_2 + \cdots + u_n) = f(u_1) + f(u_2) + \cdots + f(u_n).$$

From $f(\frac{1}{2} + 2^{-m}) = 1 - 2^{-m+1} = \frac{2}{3}(\frac{1}{2} + 2^{-m}) + \frac{2}{3}(1 - 2^{-m+2})$ it follows that for the nonzero chunks, $f(x) \geqslant \frac{2}{3}x$ with equality only for the case `11`. Obviously, for the zero chunks $f(x) = \frac{2}{3}x$. Thus:

$f(x) \geqslant \frac{2}{3}x$, with equality if and only if all the ones in $B_2(x)$ come in adjacent pairs.

It follows that $f(x) + f(1-x) \geqslant \frac{2}{3}x + \frac{2}{3}(1-x) = \frac{2}{3}$.

We now characterize the cases of equality.

If $B_2(x)$ terminates, ending in `11`, then $B_2(1-x)$ ends in `01`, so equality cannot occur.

If $B_2(x)$ does not terminate, $B_2(1-x)$ is obtained by complementing the bits of $B_2(x)$. It follows that equality occurs if and only if the ones and the zeros in $B_2(x)$ both come in adjacent pairs. Another way of putting this fact is to say that the base 4 representation $B_4(x)$ is nonterminating and contains only 0's and 3's.

An infinite family with this property is $x_k = 0.\overline{00\ldots3}_4 = 3/(4^k - 1)$. Since 3 divides $4^k - 1$, $x_k = 1/q_k$ for some odd integer k.

To complete the proof, we need to prove (1). This is clearly true when $x = 0$, and therefore also when $B_2(x)$ terminates and has an even number of ones. Since for rational x, $B_2(x)$ eventually becomes periodic, we can split $B(x)$ to obtain $x = y + 2^{-m}z$, where $B_2(y)$ has length m and has an even number of ones, $0 \leqslant z < 1$, and $B_2(z)$ is periodic with period p. Then $z = v + 2^{-2p}z$, where $B_2(v)$ is just the first $2p$ bits of $B_2(z)$, which must contain an even number of ones. It follows that $f(z) = f(v) + 2^{-2p}f(z)$, so $f(z) = f(v)/(1 - 2^{-2p})$. That is to say, $B(f(z))$ is periodic with period $2p$, repeating the bits of $B_2(f(v))$, which gives us $0 \leqslant f(z) < 2$.