The South African Mathematical Olympiad
Third Round 2008
Senior Division (Grades 10 to 12)
Solutions

1. $2008^8 = 2^{24} \times 251^8$ has $25 \times 9 = 225$ positive divisors, including $2008^4 = \sqrt{2008^8}$. There is a one-one correspondence between the positive divisors d less than 2008^4 and those larger than 2008^4, namely $d \leftrightarrow \frac{2008^8}{d}$. It follows that there are $\frac{1}{2}(225-1) = 112$ positive divisors less than 2008^4.

2. The vertices of the quadrilateral can be chosen in the xy-plane such that (without loss) $A = (a,0)$, $B = (b,0)$, $C = (0,c)$ and $D = (0,d)$, where $0 < a < b$ and $0 < d < c$. Then $AC \cdot BD > AD \cdot BC$ is equivalent to $(a^2+c^2)(b^2+d^2) > (a^2+d^2)(b^2+c^2)$ (using Pythagoras), which is equivalent to $b^2c^2 + a^2d^2 > a^2c^2 + b^2d^2$, a statement that follows immediately from the re-arrangement inequality.

3. If one of a, b, c is greater than, or equal to, the sum of the other two, then the inequality is trivially true, as the left hand side is positive while the right hand side is non-positive. Obviously, equality is not possible in this case.

   So assume that a, b, c are the lengths of the sides of a triangle. Then, using the Ravi-substitution, there are positive real numbers x, y, z such that $a = x+y$, $c = y+z$, $b = z+x$. Applying the AMG-inequality three times we get $a+b = 2x+y+z \geqslant 4(x^2yz)^{1/4}$, $b+c = x+y+2z \geqslant 4(xyz^2)^{1/4}$ and $c+a = x+2y+z \geqslant 4(xy^2z)^{1/4}$. Multiplying these three inequalities together gives

$$\begin{aligned}(a+b)(b+c)(c+a) &\geqslant 4^3(x^4y^4z^4)^{1/4}\\ &= 4^3xyz\\ &= 4^3\left(\tfrac{a+b-c}{2}\right)\left(\tfrac{b+c-a}{2}\right)\left(\tfrac{c+a-b}{2}\right)\\ &= 8(a+b-c)(b+c-a)(c+a-b).\end{aligned}$$

   Equality is obtained if and only if equality holds in all three inequalities, i.e., if and only if $x = y = z$. This happens if and only if $a = b = c$.

4. More generally, let there be n cards in the pack, numbered from 1 to n. Let $f(n)$ denote the top card before the game starts. It is easy to see that $f(n) = (n+1)/2$ if n is odd: The very first move places card $f(n)$ underneath the bottom card, say $g(n)$. The cards are then removed from the pack in the order $1, 2, \ldots, (n-1)/2$, after which $g(n)$ is now at the top and $f(n)$ is second from the top. In the next move, $g(n)$ goes underneath the pack, and $f(n)$ is removed, implying that $f(n) = (n+1)/2$.

   Now suppose that $n = 2^m k$, where $m > 0$ and k is odd. After the first $2^{m-1}k$ moves, the cards with numbers $1, 2, \ldots, 2^{m-1}k$ have been removed, and the card that was originally at the top, i.e. card no. $f(n)$, is again at the top of the remaining pack of cards with numbers $2^{m-1}k+1, 2^{m-1}k+2, \ldots, 2^{m-1}k+2^{m-1}k = 2^m k$, in some order. But this gives the recursion relation $f(n) = 2^{m-1}k + f(2^{m-1}k)$, and we immediately have

$$\begin{aligned} f(2^m k) &= 2^{m-1}k + 2^{m-2}k + \cdots + k + f(k) \\ &= (2^m - 1)k + \tfrac{k+1}{2}, \end{aligned}$$

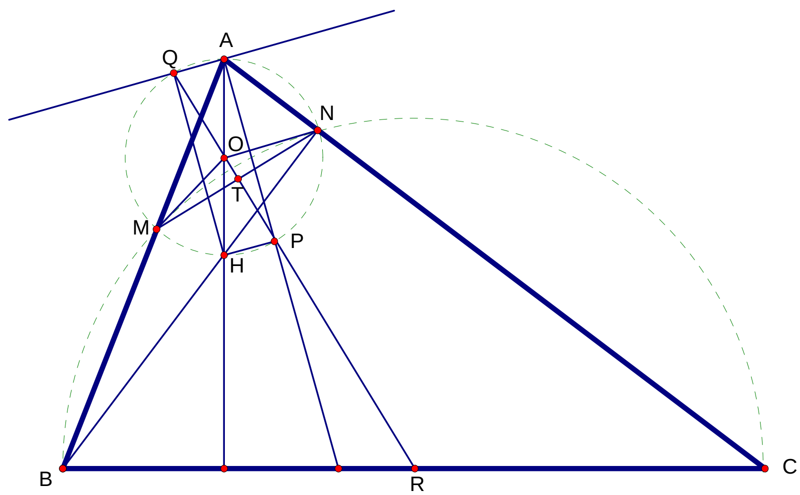
i.e., $f(n) = n - \frac{k-1}{2}$. Note that this formula also holds for odd n (if we include the possibility $m = 0$), since then $f(n) = n - \frac{n-1}{2} = \frac{n+1}{2}$.

To summarize: If $n = 2^m k$, where $m \geqslant 0$ and k is odd, then $f(n) = n - \frac{k-1}{2}$.

For $n = 2008 = 2^3 \cdot 251$, we have $f(2008) = 2008 - \frac{250}{2} = 1883$.

5. Note that if $\widehat{A} = 90^\circ$, then $P = Q$, in which case the result follows trivially.

   Henceforth, assume that $\widehat{A} \neq 90^\circ$. In the figure below, $HPAQ$ is a rectangle, since both HP and QA are perpendicular to AP, and both HQ and PA are perpendicular to QA. Join M and N, the points where the circumcircle of $HPAQ$ (with centre O) intersects AB and AC respectively.



   It follows that $M\widehat{O}P = 2M\widehat{A}P = 2N\widehat{A}P = N\widehat{O}P$, and so $\triangle MOT \equiv \triangle NOT$. This shows that, in circle $AQMHPN$, QP is a perpendicular bisector of chord MN, as it goes through the centre O, and $MT = TN$. Now since AH is a diameter of this circle, $A\widehat{N}H = 90^\circ$, showing that BN is an altitude of $\triangle ABC$. Similarly, CM is an altitude of $\triangle ABC$, and hence $BMNC$ is a cyclic quadrilateral. But then QP is a perpendicular bisector of chord MN of circle $BMNC$, so QP extended goes through the centre R of this circle, which is the midpoint of BC.

6. The functions f and g must satisfy

$$\begin{aligned} f(a+b) &= f(a)g(b) + g(a)f(b) && (1) \\ \text{and} \quad g(a+b) &= g(a)g(b) - f(a)f(b) && (2) \end{aligned}$$

   for all $a, b \in \mathbb{Z}$. By putting $a = b = 0$, we obtain

$$\begin{aligned} f(0) &= 2f(0)g(0) && (3) \\ \text{and} \quad g(0) &= g(0)^2 - f(0)^2. && (4) \end{aligned}$$

   From (3), $f(0) = 0$, otherwise $g(0) \notin \mathbb{Z}$. Hence, from (4), $g(0) = 0$ or $g(0) = 1$.

If $f(0) = g(0) = 0$, then, with a arbitrary and $b = 0$, we have (from (1) and (2)) that $f(a) = g(a) = 0$ for all $a \in \mathbb{Z}$.

Henceforth, assume that $f(0) = 0$ and $g(0) = 1$. Put

$$f(-1) = q; \quad g(-1) = p; \quad f(1) = l; \quad g(1) = k.$$

From (1) and (2) we have, for all $a \in \mathbb{Z}$,

$$\begin{aligned} f(a+1) &= kf(a) + lg(a) && (5) \\ g(a+1) &= kg(a) - lf(a). && (6) \end{aligned}$$

For $a = -1$ this becomes

$$\begin{aligned} 0 = f(-1+1) &= lp + kq && (7) \\ 1 = g(-1+1) &= kp - lq && (8) \end{aligned}$$

from which we get

$$(k^2 + l^2)q = -l. \qquad (9)$$

By (9), the integer l is a root of $qx^2 + x + qk^2 = 0$, implying that the discriminant $1 - 4k^2q^2$ is non-negative, i.e., $4k^2q^2 \leqslant 1$. So, k and q cannot be both nonzero. Let us now consider two cases:

1. $\underline{k \neq 0}$: Then $q = 0$, hence $l = 0$, by (9). From (8), $k = p = 1$ or $k = p = -1$. Consider these two subcases separately:

   1a: $\underline{k = p = 1,\ q = l = 0}$: From (5) and (6), $f(a+1) = f(a)$ and $g(a+1) = g(a)$ for all $a \in \mathbb{Z}$, i.e., both f and g are constant. So $f(1) = 0$ and $g(1) = 1$ imply that $f(a) = 0$ and $g(a) = 1$ for all $a \in \mathbb{Z}$.

   1b: $\underline{k = p = -1,\ q = l = 0}$: From (5) and (6), $f(a+1) = -f(a) = f(a-1)$ and $g(a+1) = -g(a) = g(a-1)$ for all $a \in \mathbb{Z}$, i.e., both f and g are constant on the even integers, and also constant on the odd integers. From $f(0) = 0,\ f(1) = 0,\ g(0) = 1,\ g(1) = -1$ it follows that $f(a) = 0$ and $g(a) = (-1)^a$ for all $a \in \mathbb{Z}$.

2. $\underline{k = 0}$: Then, by (8), either $q = -1$ and $l = 1$, or $q = 1$ and $l = -1$. So $p = 0$, by (7). Again, consider these two subcases separately:

   2a: $\underline{k = p = 0,\ q = 1, l = -1}$: From (5) and (6), $f(a+1) = -g(a) = -f(a-1)$ and $g(a+1) = f(a) = -g(a-1)$, i.e., both f and g alternate on the even integers, as well as on the odd integers. Hence, from $f(0) = 0,\ f(1) = -1,\ g(0) = 1,\ g(1) = 0$ it follows that $f(2a) = 0; f(2a+1) = (-1)^{a+1}$ and $g(2a) = (-1)^a; g(2a+1) = 0$ for all $a \in \mathbb{Z}$.

   2b: $\underline{k = p = 0,\ q = -1, l = 1}$: From (5) and (6), $f(a+1) = g(a) = -f(a-1)$ and $g(a+1) = -f(a) = -g(a-1)$, i.e., both f and g alternate on the even integers, as well as on the odd integers. Hence, from $f(0) = 0,\ f(1) = 1,\ g(0) = 1,\ g(1) = 0$ it follows that $f(2a) = 0; f(2a+1) = (-1)^a$ and $g(2a) = (-1)^a; g(2a+1) = 0$ for all $a \in \mathbb{Z}$.

To summarize, there are five solutions:

1. $f(a) = g(a) = 0$ for all $a \in \mathbb{Z}$.
2. $f(a) = 0$ and $g(a) = 1$ for all $a \in \mathbb{Z}$.
3. $f(a) = 0$ and $g(a) = (-1)^a$ for all $a \in \mathbb{Z}$.
4. $f(2a) = 0$; $f(2a+1) = (-1)^{a+1}$ and $g(2a) = (-1)^a$; $g(2a+1) = 0$ for all $a \in \mathbb{Z}$.
5. $f(2a) = 0$; $f(2a+1) = (-1)^a$ and $g(2a) = (-1)^a$; $g(2a+1) = 0$ for all $a \in \mathbb{Z}$.

It is easy to check that they all indeed satisfy (1) and (2).