The South African Mathematical Olympiad
Third Round 2007
Senior Division (Grades 10 to 12)
Solutions

1.

$$\begin{aligned} & & \tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{6}} &< \tfrac{3}{10} \\ \iff & & \tfrac{1}{2} - \tfrac{1}{\sqrt{3}} + \tfrac{1}{6} = \left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{6}}\right)^2 &< \tfrac{9}{100} \\ \iff & & \tfrac{173}{300} = \tfrac{2}{3} - \tfrac{9}{100} &< \tfrac{1}{\sqrt{3}} \\ \iff & & \tfrac{29929}{90000} = \left(\tfrac{173}{300}\right)^2 &< \tfrac{1}{3}, \end{aligned}$$

which is certainly true, as $29929 < 30000 = \frac{1}{3} \times 90000$. Hence $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} < \frac{3}{10}$.

2. Consider the polynomial $f(x) = x^4 - ax^3 - bx^2 - cx - 2007 = (x-\alpha)^2(x-\beta)(x-\gamma)$, where α, β, γ are distinct integers. Then, by comparing coefficients,

$$\alpha^2\beta\gamma = -2007 = -3^2 \cdot 223$$

and

$$\alpha^2 + 2\alpha(\beta+\gamma) + \beta\gamma = -b.$$

If $\alpha^2 = 1$, then $\beta\gamma = -2007$ and $-b = -2006 \pm 2(\beta+\gamma)$. In this case, the smallest value of $-b$ is obtained if either $\alpha = -1$ and $\beta+\gamma$ is as large as possible, giving $\beta+\gamma = 669 - 3 = 666$, or $\alpha = 1$ and $\beta+\gamma$ is as small as possible, giving $\beta+\gamma = -669+3 = -666$. In either case, $b = 3338$.

If $\alpha^2 = 9$, then $\beta\gamma = -223$ and $-b = -214 \pm 2\cdot 3(\beta+\gamma)$. In this case, the smallest value of $-b$ is obtained if either $\alpha = -3$ and $\beta+\gamma$ is as large as possible, giving $\beta+\gamma = 223 - 1 = 222$, or $\alpha = 3$ and $\beta+\gamma$ is as small as possible, giving $\beta+\gamma = -223+1 = -222$. In both cases, $b = 1546$.

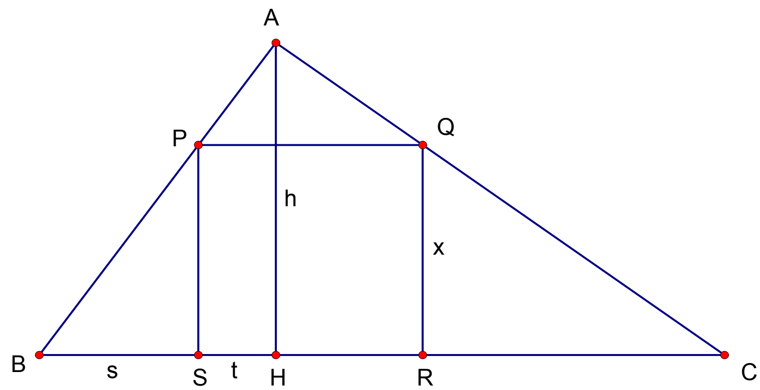
So the largest possible value for b is 3338.

3. Let $A\widehat{F}E = B\widehat{F}D = \alpha, F\widehat{D}B = E\widehat{D}C = \beta$ and $D\widehat{E}C = F\widehat{E}A = \gamma$. Note that the sum of the angles of triangle DEF is $(180^\circ - 2\alpha) + (180^\circ - 2\beta) + (180^\circ - 2\gamma) = 180^\circ$, forcing $\alpha + \beta + \gamma = 180^\circ$. Therefore, $A\widehat{C}B = \alpha$.

Now let I be the incentre of triangle DEF. Then $I\widehat{D}F = I\widehat{D}E = \beta'$, say, so that $2(\beta + \beta') = 180^\circ$, i.e., $\beta + \beta' = I\widehat{D}C = 90^\circ$. Similarly, $IE \perp AC$ and

$IF \perp AB$. Consequently, $IDCE$ and $IEAF$ are cyclic quadrilaterals, implying that $E\widehat{I}D = 180^\circ - E\widehat{C}D = 180^\circ - \alpha$, and $A\widehat{I}E = A\widehat{F}E = \alpha$. But then $A\widehat{I}D = \alpha + (180^\circ - \alpha) = 180^\circ$ and it follows that $AD \perp BC$.

4. We may assume, without loss, that angles B and C are acute: By moving A parallel to BC, the height of triangle ABC, i.e., the length of AH is unchanged, and (by the similarity between triangles APQ and ABC) the sidelength of the square doesn't change. We can therefore consider the following figure:



Put $PQ = QR = x$, $AH = h$, $BS = s$, $SH = t$, $BC = a$. Then, since $\triangle QRC \sim \triangle AHC$,

$$\frac{a - x - s}{x} = \frac{a - t - s}{h} \tag{1}$$

and, since $\triangle PSB \sim \triangle AHB$,

$$\frac{s}{x} = \frac{s + t}{h}. \tag{2}$$

Using (1) and (2), we obtain

$$\frac{a}{x} - 1 = \frac{a}{h},$$

i.e.,

$$\frac{1}{x} - \frac{1}{a} = \frac{1}{h}. \tag{3}$$

In other words,

$$\frac{1}{AH} + \frac{1}{BC} = \frac{1}{PQ}.$$

To solve (b), assume that $[ABC] = 2[PQRS]$, i.e., $\frac{1}{2}ah = 2x^2$. Then $x = \frac{1}{2}\sqrt{ah}$, so that, by (3),

$$\frac{1}{a} - \frac{2}{\sqrt{ah}} + \frac{1}{h} = 0,$$

i.e.,

$$\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{h}}\right)^2 = 0.$$

So, $\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{h}}$, giving $BC = a = h = AH$.

Conversely, if $a = h$, then $\frac{1}{x} = \frac{2}{a}$ (by (3)), so that $x = \frac{1}{2}a$. Hence, $2[PQRS] = 2x^2 = ax = \frac{1}{2}a(2x) = \frac{1}{2}ah = [ABC]$, as required.

5. (a) The function $f(n) = 0$ for all $n \in \mathbb{Z}$ satisfies the conditions trivially, and, obviously, $f(n) \leqslant 1$ for all $n \in \mathbb{Z}$ in this case.

   Henceforth, assume that there exists $m \in \mathbb{Z}$ such that $f(m) \neq 0$. By (ii), $f(m) = f(1)f(m)$, implying that $f(1) = 1$. Furthermore, for all $n \in \mathbb{Z}$, $(f(-n))^2 = f(n^2) = (f(n))^2$ (by (ii)), showing that $f(n) = f(-n)$, by (i). So f is an even function.

   From above, $f(1) = 1 \leqslant 1$. Assume that $f(n) \leqslant 1$ for some $n \geqslant 1$. Then $f(n+1) \leqslant \max\{f(n), f(1)\} = 1$, by (iii). Hence, $f(n) \leqslant 1$ for all $n \geqslant 1$, by induction, and it follows that $f(-n) = f(n) \leqslant 1$ for all $n \geqslant 1$. Finally, $f(0) = f(-1+1) \leqslant \max\{f(-1), f(1)\} = 1$.

   (b) For an integer $n \neq 0$, let $b(n) \geqslant 0$ denote the highest power of 2 that divides n. Choose a fixed real number r such that $0 < r < 1$. Then define the function f by $f(n) = r^{b(n)}$ for all $n \neq 0$. In addition, let $f(0) = 0$.

   (i) It is clear that $f(n) \geqslant 0$ for all $n \in \mathbb{Z}$.

   (ii) For nonzero m and n, we have $b(mn) = b(m)+b(n)$, so that $f(mn) = r^{b(mn)} = r^{b(m)+b(n)} = r^{b(m)}r^{b(n)} = f(m)f(n)$. If at least one of m and n is 0, it follows trivially that $0 = f(mn) = f(m)f(n)$.

   (iii) If $m+n \neq 0$ for nonzero m and n, $b(m+n) \geqslant \min\{b(m), b(n)\}$, implying that $f(m+n) = r^{b(m+n)} \leqslant r^{\min\{b(m),b(n)\}} = \max\{r^{b(m)}, r^{b(n)}\} = \max\{f(m), f(n)\}$. If $m \neq 0$ and $n = 0$, then $f(m+n) = f(m) = \max\{f(m), f(0)\}$. Finally, if $m + n = 0$, then $0 = f(m+n) \leqslant \max\{f(m), f(n)\}$, by (i).

   Since $b(2) = 1$ and $b(2007) = 0$, we have $f(2) = r$ and $f(2007) = r^0 = 1$.

6. (*Solution 1.*) We show that, for any $n \geqslant 2$, it is impossible to write the numbers $1, 2, \ldots, n^2$ on an $n \times n$ chess board such that any two neighbours differ by at most $n - 1$. The case $n = 2$ is trivial and follows immediately by inspection. Henceforth, assume that $n \geqslant 3$.

Before any numbers are written on the squares, consider a partition of the board into A-squares, B-squares and C-squares, where no A-square is a neighbour of any C-square (meaning that no A-square share a side with a C-square). This implies that the B-squares form a border between the A's and the C's. Let us assume throughout that $|A| \leqslant |C|$, where $|X|$ denotes the number of X-squares, $X \in \{A, B, C\}$. Also, the *edge* of the board consists of the squares from the top row, the bottom row, the left most column, and the right most column.

For $|B| = b$, where $2 \leqslant b \leqslant n-1$, we show that the largest possible value of $|A|$ is $\frac{b(b-1)}{2}$.

First of all, by labelling the squares of the b-th diagonal from the top left hand corner as B-squares, and letting the squares above those be A-squares, and the others C-squares, it follows that $|A| = \frac{b(b-1)}{2}$ is indeed attainable. We need to show that this bound cannot be exceeded, and this is done by induction on b. It is clear that in the case $b = 2$, $|A| \leqslant 1$ (a border of size 2 can isolate at most a corner square from the rest). Now assume that the result is true for all $|B| \leqslant b$ for some b, $2 \leqslant b < n-1$, and suppose that there is a partition for which $|B| = b+1$, and $|A| = \frac{(b+1)b}{2} + m$, for some $m > 0$.

It is possible that the A-squares form a disconnected set (i.e., there exist two A-squares such that the one is not reachable from the other by a walk through neighbours, unless a C-square is used). If a connected component (one collection of A's with its own boundary of B's) is completely surrounded by B's, it could be moved horizontally, say, to the right, until at least one of its B's is a neighbour of a B from another component, or until at least one of its B's is in the right most column of the board. Then move the component one square further to the right so that some of its B's either overlap with another component's B's, or one or more of its B-squares disappear over the edge of the board. In either case, we manage to isolate the same number of A's from a larger number of C's, but with fewer B's, and this contradicts our induction hypotheses.

The other possibility is that each component has some A-squares on the edge of the board (i.e., it is not completely surrounded by B's). Consider any one of them, with at least one B on, say, the right most column of the board. By moving the component one square to the right, we get rid of at least one B, but also some A's, say t of them, where $1 \leqslant t \leqslant b$. (Note that, in this case, no more than b A's can be situated on the right most column of the board. Also note that the component cannot have B's on two opposite sides of the board - for that you'd require $|B| \geqslant n$.) But then we obtain a new partition having a border of size at most b, for which $|A| = \frac{(b+1)b}{2} + m - t > \frac{b(b-1)}{2}$, contradicting the induction hypothesis.

To get back to the problem, suppose that it is possible to write the numbers $1, 2, \ldots, n^2$ on an $n \times n$ chess board such that any two neighbours differ by at most $n-1$. Let the A-squares be the squares with the numbers $1, 2, \ldots, \frac{n(n-1)}{2}$, the B-squares the ones with the numbers $\frac{n(n-1)}{2}+1, \ldots, \frac{n(n-1)}{2}+n-1$, and the C-squares the remaining ones. Then no A-square is a neighbour of any C-square, so the $n-1$ B-squares form a border between the A's and the C's. But since $|A| < |C|$ and $|A| = \frac{n(n-1)}{2} > \frac{(n-1)(n-2)}{2}$, we get a contradiction according to our result above. This solves the problem.

(*Solution 2, using a more direct approach in the specific case* $n = 5$.) With the view to obtain a contradiction, assume that there is a way of writing the numbers $1, 2, \ldots, 25$ on the squares of the board in such a way that all neighbours differ by at most 4.

For notational convenience, number the rows of the chess board $1, 2, \ldots, 5$ from top to bottom, and the columns $1, 2, \ldots, 5$ from left to right. (So square $(1, 1)$ is the top left hand corner, and square $(5, 5)$ is the bottom right hand corner.)

Note that if x is written on square (i, j), then the number written on square $(i+1, j+1)$ (or any of the other "diagonal neighbours") is at most $x+7$. This is because squares $(i+1, j)$ and $(i, j+1)$ contain maximal numbers $x+4$ and $x+3$, in some order.

First, the number 1 must be written on some corner square: If it has four neighbours, these neighbours must be $2, 3, 4, 5$. But then 2 needs at least two more neighbours not in this list, which is impossible. So 1 must be written somewhere on the edge of the board. If it is written on, say, square $(2, 1)$, then squares $(3, 2), (4, 3), (5, 4)$ contain maximal numbers $8, 15, 22$ respectively. So squares $(4, 5)$ and $(5, 4)$ contain maximal numbers 23 and 22 respectively. But this leaves only square $(5, 5)$ for both the numbers 24 and 25, an impossibility. If 1 is written on square $(3, 1)$, then square $(5, 5)$ contains a number not more than $1 + 2 \times 7 + 2 \times 4 = 23$, leaving no square for 25. By a dual argument (using 25 instead of 1) it follows that 25 should be written on a corner square as well. But it cannot go onto a corner adjacent to the one on which 1 is written, since then $|25 - 1| \leqslant 16$, a contradiction.

Let us assume that 1 is written on square $(1, 1)$, and 25 on square $(5, 5)$. Then the numbers $2, 3, 4$ must be written on squares from the set $\{(2, 1), (1, 2), (3, 1), (2, 2),$

$(1,3)\}$ (if x is written on square (k,l) with $k+l \geqslant 5$, then $x \geqslant 25-4[(5-k)+(5-l)] \geqslant 5$).

In particular, the number 3 must be written on either $(1,2)$ or $(2,1)$: Assume that 3 is written on square $(3,1)$. Then squares $(4,2)$ and $(5,3)$ contain maximal numbers 10 and 17 respectively, implying that squares $(4,5)$ and $(5,4)$ contain maximal numbers 22 and 21 respectively. As before, this leaves only square $(5,5)$ open for the three numbers $23, 24$ and 25, which is impossible. A similar argument applies if 3 is written on square $(1,3)$. If 3 is written on square $(2,2)$, then squares $(3,3),(4,4),(5,5)$ contain maximal numbers $10, 17$ and 24 respectively, leaving no square for 25 to be written on. As $2<3$, it cannot be written (for the same reason) on any of $(1,3),(2,2),(3,1)$, so 2 and 3 must occupy squares $(1,2)$ and $(2,1)$, in some order.

We now find that there is no room for 4 on the board. If it goes onto square $(3,1)$ or $(1,3)$, we have maximal numbers 23 and 22 on squares $(4,5)$ and $(5,4)$ (in some order), leaving, as before, only one square for both 24 and 25. If we write 4 on square $(2,2)$, it is even worse: squares $(4,5)$ and $(5,4)$ contain maximal numbers 21 and 22 (in some order), leading to the same problematic situation.

This gives us the required contradiction, and the problem is solved.