The South African Mathematical Olympiad
Third Round : 8 September 2010
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

1. For a positive integer n, $S(n)$ denotes the sum of its digits and $U(n)$ its unit digit. Determine all positive integers n with the property that
$$n = S(n) + U(n)^2.$$

2. Consider a triangle ABC with $BC = 3$. Choose a point D on BC such that $BD = 2$. Find the value of
$$AB^2 + 2AC^2 - 3AD^2.$$

3. Determine all positive integers n such that $5^n - 1$ can be written as a product of an even number of consecutive integers.

4. Given n positive real numbers satisfying $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$ and $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, prove that
$$\frac{x_1}{\sqrt{1}} + \frac{x_2}{\sqrt{2}} + \cdots + \frac{x_n}{\sqrt{n}} \geqslant 1.$$

5. (a) A set of lines is drawn in the plane in such a way that they create more than 2010 intersections at a particular angle α. Determine the smallest number of lines for which this is possible.
   (b) Determine the smallest number of lines for which it is possible to obtain exactly 2010 such intersections.

6. Write either 1 or -1 in each of the cells of a $(2n) \times (2n)$-table, in such a way that there are exactly $2n^2$ entries of each kind. Let the minimum of the absolute values of all row sums and all column sums be M. Determine the largest possible value of M.