Harmony South African Mathematics Olympiad
Third Round : 8 September 2009
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

1. Determine the smallest integer $n > 1$ with the property that $n^2(n-1)$ is divisible by 2009.

2. Let $ABCD$ be a rectangle and E the reflection of A with respect to the diagonal BD. If $EB = EC$, what is the ratio $\frac{AD}{AB}$?

3. Ten girls, numbered from 1 to 10, sit at a round table, in a random order. Each girl then receives a new number, namely the sum of her own number and those of her two neighbours. Prove that some girl receives a new number greater than 17.

4. Let $x_1, x_2, \ldots, x_n$ be a finite sequence of real numbers, where $0 < x_i < 1$ for all $i = 1, 2, \ldots, n$. Put $P = x_1 x_2 \cdots x_n$, $S = x_1 + x_2 + \cdots + x_n$ and $T = 1/x_1 + 1/x_2 + \cdots + 1/x_n$. Prove that
$$\frac{T - S}{1 - P} > 2.$$

5. A game is played on a board with an infinite row of holes labelled $0, 1, 2, \ldots$. Initially, 2009 pebbles are put into hole 1; the other holes are left empty. Now steps are performed according to the following scheme:

   - At each step, two pebbles are removed from one of the holes (if possible), and one pebble is put into each of the neighbouring holes.
   - No pebbles are ever removed from hole 0.
   - The game ends if there is no hole with a positive label that contains at least two pebbles.

   Show that the game always terminates, and that the number of pebbles in hole 0 at the end of the game is independent of the specific sequence of steps. Determine this number.

6. Let A denote the set of real numbers x such that $0 \leqslant x < 1$.
   A function $f : A \to \mathbb{R}$ has the properties:

   (i) $f(x) = 2f(\frac{x}{2})$ for all $x \in A$;
   (ii) $f(x) = 1 - f(x - \frac{1}{2})$ if $\frac{1}{2} \leq x < 1$.

   Prove that:

   (a) $f(x) + f(1 - x) \geqslant \frac{2}{3}$ if x is rational and $0 < x < 1$.
   (b) There are infinitely many odd positive integers q such that equality holds in (a) when $x = \frac{1}{q}$.