Harmony South African Mathematics Olympiad
Third Round : 4 September 2008
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

1. Determine the number of positive divisors of 2008^8 that are less than 2008^4.

2. Let $ABCD$ be a convex quadrilateral with the property that AB extended and CD extended intersect at a right angle. Prove that $AC \cdot BD > AD \cdot BC$.

3. Let a, b, c be positive real numbers. Prove that

$$(a+b)(b+c)(c+a) \geqslant 8(a+b-c)(b+c-a)(c+a-b)$$

and determine when equality occurs.

4. A pack of 2008 cards, numbered from 1 to 2008, is shuffled in order to play a game in which each move has two steps:

   (a) the top card is placed at the bottom;

   (b) the new top card is removed.

   It turns out that the cards are removed in the order $1, 2, \ldots, 2008$. Which card was at the top before the game started?

5. Triangle ABC has orthocentre H. The feet of the perpendiculars from H to the internal and external bisectors of $\widehat{A}$ are P and Q respectively. Prove that P is on the line that passes through Q and the midpoint of BC. (Note: The *orthocentre* of a triangle is the point where the three altitudes intersect.)

6. Find all function pairs (f, g), where each of f and g is a function defined on the integers and with integer values, such that, for all integers a and b,

$$\begin{aligned} f(a+b) &= f(a)g(b) + g(a)f(b); \\ g(a+b) &= g(a)g(b) - f(a)f(b). \end{aligned}$$