The South African Mathematical Olympiad
Third Round 2007
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

1. Determine whether $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}$ is less than or greater than $\frac{3}{10}$.

2. Consider the equation $x^4 = ax^3 + bx^2 + cx + 2007$, where a, b and c are real numbers. Determine the largest value of b for which this equation has exactly three distinct solutions, all of which are integers.

3. In acute-angled triangle ABC, the points D, E and F are on sides BC, CA and AB, respectively, such that $A\widehat{F}E = B\widehat{F}D$, $F\widehat{D}B = E\widehat{D}C$ and $D\widehat{E}C = F\widehat{E}A$. Prove that AD is perpendicular to BC.

4. Let ABC be a triangle and $PQRS$ a square with P on AB, Q on AC and R and S on BC (possibly extended). Let H be on BC (possibly extended) such that AH is the altitude of the triangle from A to the base BC. Prove that:
   (a) $\frac{1}{AH} + \frac{1}{BC} = \frac{1}{PQ}$;
   (b) the area of ABC is twice the area of $PQRS$ if and only if $AH = BC$.

5. Let $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of integers and real numbers, respectively.
   Let $f : \mathbb{Z} \to \mathbb{R}$ be a function satisfying:
   (i) $f(n) \geqslant 0$ for all $n \in \mathbb{Z}$;
   (ii) $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}$;
   (iii) $f(m + n) \leqslant \max\{f(m), f(n)\}$ for all $m, n \in \mathbb{Z}$.

   (a) Prove that $f(n) \leqslant 1$ for all $n \in \mathbb{Z}$.
   (b) Find a function $f : \mathbb{Z} \to \mathbb{R}$ satisfying (i), (ii) and (iii) that also satisfies $0 < f(2) < 1$ and $f(2007) = 1$.

6. Show that it is not possible to write the numbers $1, 2, \ldots, 25$ on the squares of a 5×5 chess board (one number per square) such that any two neighbouring numbers differ by at most 4. (Two numbers are *neighbours* if they are written on squares that share a side.)