THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior Second Round 2008

Solutions

1. **Answer A.** The product includes factors of 4 and 25, so it is divisible by 100, so the tens digit is 0. [In fact, the last 22 digits are all zero.]

2. **Answer A.** The unused numbers in the fourth column are 1, 2 and 3. The third row already contains 1 and 2, so 3 must go above 4. This leaves 1 and 2 as possibilities for x, but since the fifth row already contains 2, it follows that $x = 1$.

3. **Answer C.** If the radius of the small circle is r, then the radius of the big circle is $2r$. The ratio of the areas is $\dfrac{\pi(2r)^2}{\pi r^2} = 4$.

4. **Answer E.** $\sqrt{36^{36}} = \sqrt{6^{2\times 36}} = 6^{2\times 36/2} = 6^{36}$.

5. **Answer B.** The remainder when any number is divided by 3 is equal to the remainder when the sum of its digits is divided by 3. Here the sum of the digits is 10, which leaves remainder 1 after being divided by 3.

6. **Answer C.** The single-digit pages 1 to 9 use up 9 digits, leaving $173 - 9 = 164$ digits for the two-digit pages. There must therefore be $164 \div 2 = 82$ two-digit pages, so the book contains a total of $9 + 82 = 91$ pages.

7. **Answer A.** If a and b are the two roots of the equation, then $x - a$ and $x - b$ are factors of the quadratic. Thus $x^2 + ax + b = (x - a)(x - b)$, giving $x^2 + ax + b = x^2 - (a + b)x + ab$. It follows that $b = ab$ and $a = -a - b$. Since $b \neq 0$, we have $a = 1$ and $b = -2a = -2$.
   Alternatively, substitute $x = a$ and $x = b$ in the equation to get $2a^2 + b = 0$ and $b^2 + ab + b = 0$, so $b = -2a^2$ and $b = -a - 1$. (We can cancel b since $b \neq 0$.) It follows that $2a^2 - a - 1 = 0$. One solution is $a = 1$ and $b = -2$, as before. However, this also gives the spurious solution $a = b = -\frac{1}{2}$, which is only one root of the equation $x^2 - \frac{1}{2}x - \frac{1}{2} = 0$.

8. **Answer D.** We have $10 < \sqrt{n} < 12$, so $100 < n < 144$, which means that n can be any one of the 43 numbers from 101 to 143 inclusive.

9. **Answer E.** For each corner,there are two other corners at a distance 1 and one other corner at a distance $\sqrt{2}$. The average distance is therefore equal to $(2 \times 1 + 1 \times \sqrt{2})/3$.

10. **Answer B.** If the side is of length x, then by Pythagoras' theorem $x^2 + (\frac{1}{2}x)^2 = 1^2$, giving $\frac{5}{4}x^2 = 1$. The area of the square is equal to $x^2 = \frac{4}{5}$.

11. **Answer D.** We are given $S = s^2$ and $D = \frac{1}{2}d \times \frac{1}{2}\sqrt{3}d = \frac{1}{4}d^2\sqrt{3}$. Thus

$$\sqrt{3} = \frac{D}{S} = \frac{\frac{1}{4}d^2\sqrt{3}}{s^2} = \frac{d^2\sqrt{3}}{4s^2}.$$

   It follows that $d^2 = 4s^2$, so $d = 2s$.

12. **Answer C.** Each of the five lines can intersect with each of the other four lines, giving a total of 20. However, this counts each intersection twice, since the order of the lines does not matter, so the total number of intersections is 10. [This is the number of ways of choosing two objects out of five, and is sometimes written $\binom{5}{2}$.]
Alternatively, two lines have at most one intersection, then a third line can produce two more intersections, a fourth line three more, and a fifth line four more intersections. Thus the maximum possible is $1 + 2 + 3 + 4 = 10$.

13. **Answer C.** We can approximate a regular polygon with 2008 sides by a circle. A circle with perimeter 1 has radius $\dfrac{1}{2\pi}$ and area $\dfrac{\pi}{4\pi^2} = \dfrac{1}{4\pi} \approx \dfrac{1}{12}$.

14. **Answer B.** A number is divisible by 9 if and only if the sum of its digits is divisible by 9, so we need to find three odd digits whose sum is 9 or 27. (The sum of three odd numbers is odd, so 18 is impossible.) The possibilities for the digits are: (a) $1, 1, 7$, which can be arranged to give three different numbers, (b) $1, 3, 5$, which can be arranged to give six different numbers, (c) $3, 3, 3$, which gives only one number, (d) $9, 9, 9$, which also gives one number. The total is therefore $3 + 6 + 1 + 1 = 11$.

15. **Answer A.** Substitute $x = 1$ to give $f(2) + f(\frac{1}{2}) = 1$ and substitute $x = -1$ to give $f(\frac{1}{2}) - f(2) = 1$. Now subtract the two equations to obtain $2f(2) = 0$, so $f(2) = 0$.

16. **Answer B.** At the first stage there is one semicircle, and at the second stage we have two semicircles with half the original diameter, and therefore one-quarter of the area (see Question 3). The two grey semicircles therefore make up $2 \times \frac{1}{4} = \frac{1}{2}$ of the first semicircle. Similarly, at the next stage the four white semicircles contain $4 \times \frac{1}{16} = \frac{1}{4}$ of the original area. At each stage there are twice as many semicircles, each with one-quarter of the area of the previous ones, so their total area is halved. Since the two colours alternate, we need to add and subtract the areas alternately in order to determine the eventual white area. As a fraction of the original, the white area is therefore equal to

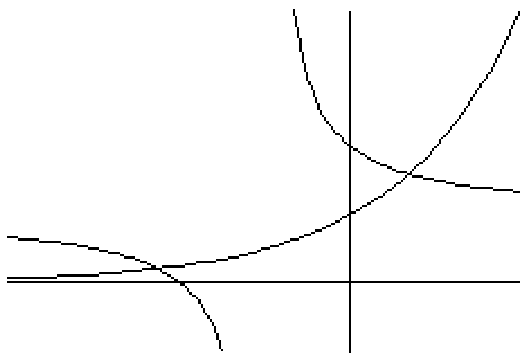
$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \cdots = \frac{1}{1 - (-1/2)} = \frac{2}{3}.$$

17. **Answer D.** Each rectangle is uniquely defined by its four sides, that is, by two vertical lines and two horizontal lines. For each rectangle we need to choose two out of the five vertical lines, which can be done in 10 ways, as in Question 12, and also choose two out of the five horizontal lines, also in 10 ways. By combining each vertical choice with each horizontal choice we obtain a total of $10 \times 10 = 100$ rectangles.

18. **Answer E.** Join the top vertex to the point of intersection of the two lines in the triangle, thus dividing the area x into two sub-areas x_1 and x_2. We use the fact that if the triangles have the same height, then their areas are proportional to their bases. Using the left hand side of the triangle as the base, we see that $\dfrac{8}{x_1} = \dfrac{18}{x + 5}$, so $8(x_1 + x_2) + 40 = 18x_1$, giving $5x_1 - 4x_2 = 20$. Next, using the right hand side of the triangle as the base, we see that

$\dfrac{5}{x_2} = \dfrac{15}{x+8}$, so $5(x_1 + x_2) + 40 = 15x_2$, giving $x_1 - 2x_2 = -8$. Solve the equations to get $x_1 = 12$ and $x_2 = 10$, and finally $x = x_1 + x_2 = 22$.

19. **Answer C.** The equation can be rewritten as $2^x = 1 + \frac{1}{x+1}$, so solutions of the equation correspond to intersections of the graphs $y = 2^x$ and $y = 1 + \frac{1}{x+1}$. The exponential graph $y = 2^x$ is always positive, has the negative x-axis as an asymptote, cuts the y-axis at $y = 1$ and is always increasing. The graph $y = 1 + \frac{1}{x+1}$ is a rectangular hyperbola, with its asymptotes shifted to the lines $y = 1$ (horizontal) and $x = -1$ (vertical). A sketch shows that the exponential graph cuts each branch of the hyperbola exactly once, giving a total of two intersection points, and therefore two solutions of the original equation. Alternatively, by looking at the difference $2^x - 1 - \frac{1}{x+1}$ at suitably chosen points, we can see that it changes sign between -3 and -2 and again between 0 and 1. It follows that the equation has a solution in each of these intervals. However, it does not guarantee that there are no other solutions.



20. **Answer D.** Firstly, the first term must be odd in order to ensure that the sum of the four terms is even, so $a = 0$, and we are left with $5^b + 7^c + 11^d = 2005$. Next 5^b has last digit 5 (unless $b = 0$, which we can come back to if necessary) and 11^d has last digit 1, so 7^c must have last digit 9 to get a sum of 2005. It follows that $c = 2$ and $7^c = 49$. Subtracting this from both sides gives $5^b + 11^d = 1956$. The largest powers of 5 and 11 that are less than 1956 are $5^4 = 625$ and $11^3 = 1331$, and their sum is 1956, as required. Thus $b = 4$ and $d = 3$, so $a + b + c + d = 0 + 4 + 2 + 3 = 9$.
