THE HARMONY SOUTH AFRICAN MATHEMATICS OLYMPIAD

Organised by the SOUTH AFRICAN MATHEMATICS FOUNDATION
Sponsored by HARMONY GOLD MINING

SECOND ROUND 2007

SENIOR SECTION: GRADES 10, 11 AND 12

15 MAY 2007

TIME: 120 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. Answer D
2. Answer C
3. Answer A
4. Answer B
5. Answer D
6. Answer D
7. Answer D
8. Answer E
9. Answer B
10. Answer A
11. Answer E
12. Answer E
13. Answer D
14. Answer C
15. Answer C
16. Answer E
17. Answer A
18. Answer A
19. Answer D
20. Answer A

SOLUTIONS

1. **Answer D.** If 8000 die every day, then the number that die every hour is $8000/24 \approx 333$, and the number that die every two minutes is approximately $333/30 \approx 11$.

2. **Answer C.** $1\nabla(2\nabla 3) = 1\nabla(\frac{2}{2+3}) = 1\nabla(\frac{2}{5}) = \frac{1}{1+(2/5)} = \frac{1}{7/5} = \frac{5}{7}$.

3. **Answer A.** The original piece of paper has sides L and W, where L = Length and W = Width. After folding, the sides are W and $\frac{1}{2}L$. Since we are given that $\frac{L}{W} = \frac{W}{(\frac{1}{2}L)}$, it follows that $L^2 = 2W^2$, so $\frac{L}{W} = \sqrt{2}$ (since both are positive).

4. **Answer B.** If the diameter is 80 cm, then the circumference is 80π cm ≈ 250 cm $= 2.5$ m. The distance travelled is 120 km $= 120\,000$ m. The number of revolutions is therefore about $120\,000/2.5 \approx 48\,000$, and $50\,000$ is the nearest answer supplied.

5. **Answer D.** Since $b = a\sqrt{a}$, and b is an integer, it follows that a must be a perfect square and b must be a perfect cube. The possibilities for b are $1^3 = 1$, $2^3 = 8$, $3^3 = 27$, and $4^3 = 64$.

6. **Answer D.** The triangles PCD and PBA are similar, so the ratio of their heights is equal to $CD/AB = 12/4 = 3$. Thus if triangle PCB has height h, then trangle PBA has height $\frac{1}{3}h$. The distance between the lines is 8, so $h + \frac{1}{3}h = 8$, giving $h = 6$.

7. **Answer D.** Given $3^x = 2$, consider (A): $x < \frac{3}{4} \iff 3^x < 3^{3/4} \iff 2 < 3^{3/4} \iff 2^4 < 3^3 \iff 16 < 27$, which is true. Similarly, (B): $x > \frac{4}{7} \iff 2^7 > 3^4 \iff 128 > 81$ and (C): $x < \frac{2}{3} \iff 2^3 < 3^2 \iff 8 < 9$, which are also both true. However, (D): $x < \frac{5}{8} \iff 2^8 < 3^5 \iff 256 < 243$, which is false. As a check, consider (E): $x > \frac{3}{5} \iff 2^5 > 3^3 \iff 32 > 27$, which is true, as expected. [The symbol $\iff$, read "if and only if", links two statements that are logically equivalent, which means that if either statement is true then so is the other.] By arranging the values in order, and knowing that exactly one statement is false, you can eliminate some of the options by pure logic.

8. **Answer E.** Suppose the fraction is $\frac{x}{x+16}$. We are given that $\frac{5}{9} < \frac{x}{x+16} < \frac{4}{7}$, so $5x+80 < 9x$ and $7x < 4x+64$. It follows that $20 < x < 21\frac{1}{3}$, and the only possible integer value for x is $x = 21$.

9. **Answer B.** There are 500 voters, of whom 40 are against both issues, leaving 460 who vote in favour of at least one issue. If there are x voters in favour of both issues, then $275 + 375 - x = 460$, so $x = 190$.

10. **Answer A.** Of the 49 possible choices for x and y, we must exclude those for which $x+y < 5$ (because then $z > 7$) and those for which $x+y > 11$ (because then $z < 1$). There are $1+2+3$ choices with $x+y = 2, 3, 4$, respectively, and $3+2+1$ choices with $x+y = 12, 13, 14$, respectively. This leaves a total number of $49 - 6 - 6 = 37$ solutions.

11. **Answer E.** By clearing fractions, we see that $ab = 6(a+b)$, which can be rewritten as $(a-6)(b-6) = 36$. From the factorizations $36 = 1\times 36, 2\times 18, 3\times 12, 4\times 9, 6\times 6$, respectively, we obtain the solutions $(a,b) = (7,42), (8,24), (9,18), (10,19), (12,12)$, remembering that $a \leq b$.

12. **Answer E.** Since the bridge MN has length 1 wherever M and N may be, we only need to find the shortest length for $AM + NB$. The easiest way is to forget about the river and make M coincide with N. The shortest distance of $AM + MB$ occurs when AMB is a straight line. In this case, AMB is the hypotenuse of a right-angled triangle whose other sides are 5 and 12, so $AMB = 13$. If we now replace the river, then $AM + NB = 13$ and $MN = 1$, so the total distance is 14.

13. **Answer D.** By multiplying out the factored form and equating coefficients, we obtain the equations $a+b+c = a$, $ab+bc+ca = b$, $abc = c$. These can be simplified to $b + c = 0$, $bc = b$, $ab = 1$, since a, b, c are non-zero.The only solution is $(a,b,c) = (-1,-1,1)$, so $p(2) = (2+1)(2+1)(2-1) = 9$. [Alternatively, by substitution in the factored form, we see that $p(a) = p(b) = p(c) = 0$. These lead to rather more complicated equations in a, b, c.]

14. **Answer C.** In the figure there are several similar right-angled triangles with sides in the ratio $2 : 3 : \sqrt{13}$. Since the shaded square has side 1, it follows that $AE = \dfrac{\sqrt{13}}{3}$, so $AB = \sqrt{13}$, and the area of the square $ABCD$ is 13.

15. **Answer C.** Since April has 30 days, the $2k$-th day of May can be thought of as the $(2k+30)$-th day of April, so it is $k+30$ days after the k-th day of April. Thus $k+30$ must be divisible by 7, which happens when $k = 5,\ 12,\ 19,\ 26$. We have to discard the last two values since $2k > 31$ if $k = 19$ or $k = 26$. [In the language of modular arithmetic, we have $k \equiv 2k + 30 \bmod 7$, since the difference between k and $2k + 30$ is divisible by 7.]

16. **Answer E.** The combined volume of the six barrels is 119 litres. We must eliminate one barrel so that the combined volume of the remaining barrels is divisible by 3. [In other words, we must solve $x \equiv 119 \bmod 3$, where x litres is the volume of one of the barrels.] By inspection, the solution is $x = 20$.

17. **Answer A.** With no 7-cent stamps, we can pay amounts of

$$6, \ldots, 12, \ldots, 18, \ldots, 24, \ldots, 30, \ldots, 36, \ldots,$$

with five blank spaces in between successive possible values. To fill the five blanks, we need up to five 7-cent stamps, so from 35 onwards all the spaces can definitely be filled. Check backwards to see that the largest blank value is 29.

18. **Answer A.** We must build up to $(x, y) = (12, 5)$ by substituting appropriate values for x and y. First, $f(12, 0) = 12 = f(0, 12)$. Then $f(1, 12) = f(0, 12) + 12 + 1 = 25$, and $f(2, 12) = f(1, 12) + 12 + 1 = 38$, and so on (adding 13 each time) until we see that $f(5, 12) = 77$, which is the same value as $f(12, 5)$.

19. **Answer D.** The sum of the angles of a pentagon (five-sided figure) is $(5-2)180° = 540°$. If the smallest angle (in degrees) is a and the difference between consecutive angles is d, then $5a + 10d = 540$, so $a + 2d = 108$, giving $a = 2(54 - d)$. For a convex pentagon we also need that the fifth angle should be less than 180° which means that $a + 4d < 180$. Since $a = 2(54 - d)$ it follows that $2(54 - d) + 4d < 180$, or $d < 36$. This gives 36 possible integer values for d from $d = 0$ to $d = 35$ (note that a cannot equal zero, but d can), each of which gives a unique value for a.

20. **Answer A.** The triangle is right-angled, since its longest side is a diameter of the large circle. Drop perpendiculars of length $\frac{1}{2}d$ from the centre of the small circle to the three sides of the triangle. The distances from the right-angle to the points of contact are equal to $\frac{1}{2}d$, and the distances from the other two angles to the points of contact are $a - \frac{1}{2}d$ and $b - \frac{1}{2}d$, so $a - \frac{1}{2}d + b - \frac{1}{2}d = D$, giving $D + d = a + b$.