THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior First Round 2011

Solutions

1. **Answer B.**
$0.125 + \frac{3}{4} = 0.125 + 0.75 = 0.875$. Alternatively, $0.125 + \frac{3}{4} = \frac{1}{8} + \frac{3}{4} = \frac{1}{8} + \frac{6}{8} = \frac{7}{8} = 0.875$.

2. **Answer C.**
$1\ 000\ 000 \div 100 = 10\ 000$, which must equal $100\ 000 \div x$, where x denotes the unknown number. Therefore, $x = 100\ 000 \div 10\ 000 = 10$. Alternatively, using exponents and writing $x = 10^n$, we have $10^6 \div 10^2 = 10^{6-2} = 10^4$ on the left hand side and $10^5 \div 10^n = 10^{5-n}$ on the right. Thus $4 = 5 - n$ (since the exponents are equal), so $n = 5 - 4 = 1$ and $x = 10^1 = 10$.

3. **Answer B.**
Use dimensions of cm. If each edge of the cube has length x, then its surface area is $6x^2$. (The cube has six square faces with sides of length x.) Thus $6x^2 = 54$, so $x^2 = 9$ and $x = 3$. The volume of the cube is $x^3 = 3^3 = 27\,\text{cm}^3$.

4. **Answer B.**
Since the number of balls of each colour has to be a whole number, it follows that the total number of balls in the box must be divisible by 3, 4, and 6. The least common multiple of these three numbers (that is, the least number divisible by all of them) is 12.

5. **Answer E.**
When two boxes are put on the balance beam, the lighter box goes up. Thus we see from the figure that $A < E$, $C < D$, $B < A$, and $E < C$. Arranged in order, these inequalities are $B < A < E < C < D$, and the one in the middle is E.

6. **Answer B.**
Suppose the digits of the number are HTU; then $5 \leq H \leq 9$ (since the number lies between 500 and 900) and $H + U = T^2$. Now, $H + U$ lies between $5 + 0 = 5$ and $9+9 = 18$ and it has to be a perfect square, since it equals T^2. The only possibilities are 9 and 16, so $T = 3$ or $T = 4$. Now list the possible numbers in order: firstly, with $T = 3$ and $H + U = 9$, we have 534, 633, 732, and 831. (Note that 930 is too large.) Secondly, with $T = 4$ and $H + U = 16$, we have 749 and 848. (If $H < 7$, then $U > 9$, which is impossible for a single digit, and if $H = 9$, then the number is 947, which is too large.) Thus there are exactly six possible numbers.

7. **Answer A.**
The ant's path is equal to five edges of the cube (the portions at the bottom and top of the front face together make up one edge), so the total length in cm is $5 \times 12 = 60$.

8. **Answer E.**
If we cut off the $a \times a$ square on the lower right, and insert it into the $a \times a$ gap in the left wall, we get a rectangle with dimensions $3a$ and b, with area $3ab$.

9. **Answer A.**
On each flip of the coin, it must come up either heads (H) or tails (T), each with probability $\frac{1}{2}$. For three tosses, there are eight equally likely outcomes, namely

$$HHH, HHT, HTH, THH, HTT, THT, TTH, TTT,$$

each with probability $\frac{1}{8}$. Three of these (HTT, THT, TTH) have exactly one head, so the probability of one head is $\frac{3}{8}$.

10. **Answer C.**
Each child is weighed three times (once with each of the other three children), so the sum of the recorded masses is three times the total mass of the children. Thus the total mass is $(85 + 92 + 95 + 97 + 100 + 107) \div 3 = 576 \div 3 = 192$ kg. One can also get the total mass by adding $85 + 107 = 192$ kg, because the 85 kg corresponds to the two lightest children and the 107 kg to the two heaviest children.

11. **Answer A.**
Since 2 and 5 are included in the first 100 prime numbers, it follows that $M = 2\times 5\times$ the other 98 primes. Thus M is a multiple of 10, so its last digit is 0.

12. **Answer B.**
The largest 5-digit number with no repeated digits is 98765, since the largest digits are in the highest places. The smallest such number is not 01234 (because this is a 4-digit number) but in fact 10234. The difference is $98765 - 10234 = 88531$.

13. **Answer D.**
Suppose rectangle R has length L and width W, so its area is LW. Then rectangle S has length $\dfrac{110}{100}L = \dfrac{11}{10}L$ and it has width $\dfrac{90}{100}W = \dfrac{9}{10}W$. Thus the area of rectangle S is $\dfrac{11}{10}L \times \dfrac{9}{10}W = \dfrac{99}{100}LW$, which is 99% of the area of rectangle R, that is, 1% less.

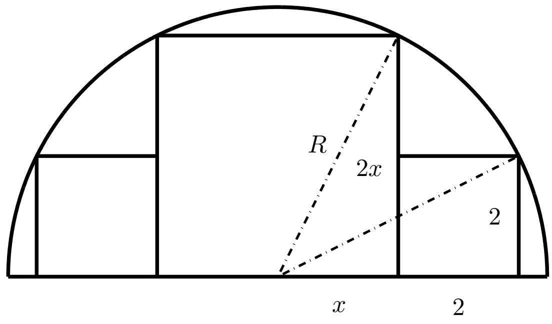
14. **Answer A.**
The sum of the numbers on all seven cards is $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$. One number on the left hand side is removed, and the units digit of the sum is then 7. Thus the new sum must be 27 (because to get 17 or less you would need to remove 11 or more, which is impossible), and the drawn card must be numbered 1.

15. **Answer D.**
Since the sum of the angles of a quadrilateral is 360°, it follows that the angle at the point of each kite is $360° - 2\theta - \theta - 2\theta = 360° - 5\theta$. (By symmetry, both angles next to angle θ are equal to 2θ.) The points of the eight kites meet at the centre of the figure and make up a complete revolution, which is 360°. Thus $8(360° - 5\theta) = 360°$, or $360° - 5\theta = 45°$, giving $\theta = \dfrac{315°}{5} = 63°$.

16. **Answer B.**
One method (multiple-choice technique) is to square the given numbers until you find one whose square is equal to $4 - 2\sqrt{3}$ and which is positive. (Remember that a square root cannot be negative.) A more direct way is to suppose $\sqrt{4 - 2\sqrt{3}} =$

$a + b\sqrt{3}$, where a and b are rational numbers. Squaring both sides gives $4 - 2\sqrt{3} = a^2 + 2ab\sqrt{3} + 3b^2$. We can now equate rational and irrational parts to give $4 = a^2 + 3b^2$ and $-2\sqrt{3} = 2ab\sqrt{3}$, so $ab = -1$. By inspection, or by substituting $b = -1/a$ into $a^2 + 3b^2 = 4$, we see that $a = \pm 1$ and $b = \mp 1$, so the full solution is $\pm(1 - \sqrt{3})$. We must choose the positive one, which is $-1 + \sqrt{3}$.

17. **Answer C.**
Let R be the radius of the semicircle, and suppose the large square has sides of length $2x$. By applying Pythagoras' theorem twice we see that $x^2 + (2x)^2 = R^2$, and $(x + 2)^2 + 2^2 = R^2$. Therefore $5x^2 = x^2 + 4x + 8$, so $4x^2 - 4x - 8 = 0$ or $x^2 - x - 2 = 0$. The solutions are $x = 2$ and $x = -1$, but $x = -1$ is impossible, since the side length must be positive. Finally, the area of the semicircle is $\frac{1}{2}\pi R^2 = \frac{1}{2}\pi 5x^2 = 10\pi$.



18. **Answer D.**
The shaded region is obtained by removing the quarter circle from the combined triangle and semicircle. The triangle has area $\frac{1}{2} \times 6 \times 8 = 24$. By Pythagoras' theorem, $AC = \sqrt{6^2 + 8^2} = 10$. Thus the semicircle has radius 5 and therefore area $\dfrac{1}{2}\pi 5^2$. The quarter circle has area $\dfrac{1}{4}\pi 6^2$. Therefore, the area of the shaded region is $24 + \dfrac{25}{2}\pi - 9\pi = 24 + \dfrac{7}{2}\pi$. For the specified accuracy it is more than sufficient to take π as $22/7$, which yields the area as approximately $24 + 11 = 35$.

19. **Answer B.**
Suppose the distance between the towns is x km. The first time the cars pass each other, the first (from town A) has gone 70 km and the other has gone $(x - 70)$ km. These distances were covered in the same time, so the ratio of the speeds of the cars is $\dfrac{70}{x - 70}$. The next time they pass the first has travelled $(x+40)$ km and the second has travelled $(2x - 40)$ km. Thus the ratio of the speeds is also equal to $\dfrac{x + 40}{2x - 40}$. We can equate the two ratios to get $\dfrac{70}{x - 70} = \dfrac{x + 40}{2x - 40}$, so $70(2x - 40) = (x + 40)(x - 70)$. This gives $140x - 2800 = x^2 - 30x - 2800$, or $x(x - 170) = 0$. The solution $x = 0$ is meaningless, so the towns are 170 km apart.

20. **Answer E.**
Method 1: Let

$$S = 1 + 11 + 111 + 1111 + \cdots + \underbrace{111\ldots111}_{\text{2011 digits}}$$

Then

$$10S = 0 + 10 + 110 + 1110 + \cdots + \underbrace{111\ldots110}_{\text{2011 digits}} + \underbrace{111\ldots110}_{\text{2012 digits}}$$

Subtract

$$9S = \underbrace{-1-1-1-1\cdots-1}_{\text{2011 numbers}} + \underbrace{111\ldots110}_{\text{2012 digits}} = \underbrace{111\ldots110}_{\text{2012 digits}} -2011 = \underbrace{111\ldots11109099}_{\text{2012 digits}}.$$

We want to divide by 9 to obtain S. Notice that there are 2007 ones before the 09099. Now $2007 = 223 \times 9$, so we can regard the 2007 ones as made up of 223 consecutive blocks each consisting of 9 ones. Next notice that 111111111 divided by 9 gives us 012345679. Also 09099 divided by 9 gives us 01011. So S consists of 012345679 repeated 223 times followed by 01011. Therefore there are $223+3=226$ ones.

Method 2: Write S as

$$S = \frac{10^0-1}{9} + \frac{10^1-1}{9} + \frac{10^2-1}{9} + \cdots + \frac{10^{2011}-1}{9}$$

and therefore

$$9S = 10^0 + 10^1 + 10^2 + \cdots + 10^{2011} - 2012 = \underbrace{111\ldots111}_{\text{2012 digits}} -2012 = \underbrace{111\ldots11109099}_{\text{2012 digits}}$$

From here the argument is the same as in *Method 1*.