THE SOUTH AFRICAN MATHEMATICS OLYMPIAD
SENIOR FIRST ROUND 2010
SOLUTIONS

1. **Answer A.** On 11 June the count-down clock shows 0 days, so on 1 June it shows 10 days, on 1 May it shows $10+31 = 41$ days, on 1 April it shows $41+30 = 71$ days, and on 17 March it shows $71 + 15 = 86$ days. Alternatively, counting from 1 March we see that 31 March is day 31, 30 April is day 61, 31 May is day 92, and 11 June is day 103. The number of days from 17 March is $103 - 17 = 86$.

2. **Answer C.** We can cancel 2.010 twice and see that the fraction is equal to $\frac{1\times1000}{10\times100} = \frac{1000}{1000} = 1$.

3. **Answer D.** The average of the five numbers is $50 \div 5 = 10$, so the middle number is 10. The five numbers are therefore 8, 9, 10, 11 and 12.

4. **Answer B.** If 7 cm represents 56 km, then 1 cm represents $56 \div 7 = 8$ km, and 19 cm represents $19 \times 8 = 152$ km

5. **Answer E.** $2^{-2} + 2^{-6} = \frac{1}{4} + \frac{1}{64} = \frac{16}{64} + \frac{1}{64} = \frac{16+1}{64} = \frac{17}{64}$.

6. **Answer B.** Aaron Mokoena runs 12 km in 90 minutes, which is 1.5 hours, so his average speed is $12 \div 1.5 = 8$ km/h.

7. **Answer D.** Suppose there are n chickens, which together have n heads and $2n$ legs. There are also $2n$ sheep, which have $2n$ heads and $8n$ legs. Thus $n + 2n + 2n + 8n = 91$, so $13n = 91$ and $n = 91 \div 13 = 7$.

8. **Answer D.** Working in metres, let one side of the square lawn be x, then one side of the outer square (including the path on each side) will be $x + 2$. The area of the path is then $(x+2)^2 - x^2 = 4x+4$, which is equal to 40. Therefore $4x+4 = 40$, so $x = (40-4) \div 4 = 9$ and the area of the lawn is $x^2 = 9^2 = 81$.

9. **Answer E.** Cost price was Rx. Selling price is $x+\frac{1}{4}x = \frac{5}{4}x$. Reduced price is $\frac{5}{4}x - \frac{1}{5} \times \frac{5}{4}x = x$. So the percentage change is zero.

10. **Answer A.** P(red)=$\frac{1}{4}$. P(yellow)=$\frac{1}{3}$. P(blue)=$1 - \frac{1}{4} - \frac{1}{3} = \frac{5}{12}$. The least number of blue Smarties is 5 (with 3 red ones and 4 yellow ones).

11. **Answer E.** Use the additive results Odd $\pm$ Odd = Even, Odd $\pm$ Even = Odd, Even $\pm$ Even = Even, and the multiplicative results Odd $\times$ Odd = Odd, Odd $\times$ Even = Even, Even $\times$ Even = Even to show that all expressions are odd except for $p^2 - p$, which is Odd $-$ Odd or, in the form $p(p-1)$ it is Odd $\times$ Even.

12. **Answer A.** If the cube has side length x cm, then $7x^3 = 189$, so $x^3 = 189 \div 7 = 27$ and therefore $x = 3$. One cube is completely hidden, and the remaining six cubes each have five faces on the surface of the solid. Therefore the surface area is $6\times5\times x^2 = 30\times9 = 270$ cm^2.

13. **Answer C.** Working in degrees, let $\widehat{A} = x$; then $F\widehat{C}A = x$ since $AF = FC$. Next $C\widehat{F}B = 2x$ (exterior angle), so $\widehat{B} = 2x$ also, since $FC = CB$. Thus $F\widehat{C}B = 180 - 4x$ (angle sum of triangle), so $A\widehat{C}B = x + 180 - 4x = 180 - 3x$. But $A\widehat{C}B = \widehat{B}$ since $AB = AC$, so $180 - 3x = 2x$. Therefore $5x = 180$ so $x = 36$.

14. **Answer D.** If the numbers are x and y, then we are given that $x + y = m$ and $xy = n$. Now $(x + y)^2 = x^2 + 2xy + y^2$, so $x^2 + y^2 = (x + y)^2 - 2xy = m^2 - 2n$.

15. **Answer B.** In a rhombus all four sides are equal and the diagonals divide the rhombus into four congruent right-angled triangles. If the longer diagonal is $2x$, then the shorter diagonal is $2x - 8$, and the area of each triangle is $\frac{1}{2}x(x-4)$. This is one-quarter of the area of the rhombus, so $\frac{1}{2}x(x-4) = \frac{1}{4} \times 24 = 6$, giving $x(x-4) = 12$. By inspection we see that $x = 6$, so the length of the diagonal is $2x = 2 \times 6 = 12$. Otherwise the quadratic equation $x^2 - 4x - 12 = 0$ factorises as $(x-6)(x+2) = 0$, so $x = 6$ or $x = -2$, and clearly the negative solution is meaningless.

16. **Answer D.** We can write $x = (3-1) + (7-5) + \cdots + (199-197)$. Each bracket is equal to 2 and there are 50 brackets (count carefully), so $x = 2 \times 50 = 100$.

17. **Answer A.** We need to have $N + S = E + W$, since M appears on both sides. There are three possible equations with the given numbers: $2+3 = 1+4$, $2+4 = 1+5$, and $2+5 = 3+4$. For each of these equations there are four choices for N (any one of the four numbers in the equation) and two choices for E (one of the two numbers on the other side of the equation). Now S, W and M are fixed, so there are 4×2 arrangements for each of the three equations. The total number of arrangements is therefore $3 \times 8 = 24$. Alternatively, with each of the three equations fill in one possible arrangement for the numbers. There are four arrangements (including the first) obtainable by rotating the figure, and another four obtainable by reflecting the figure. This gives eight arrangements for each of the three equations, giving a total of 24 arrangements.

18. **Answer B.** $2^{2009} + 2^{2010} = 2^{2009}(1+2) = 3 \times 2^{2009}$. If we look at the first few remainders of 3×2^n after dividing by 5 we obtain the values

n:	0	1	2	3	4	5
3×2^n:	3	6	12	24	48	96
Remainder:	3	1	2	4	3	1

The remainders repeat in a cycle of length 4. Since $n = 2009$ is of the form $n = 4k+1$, the remainder is the same as for $n = 1$ and $n = 5$, so the remainder is 1.

19. **Answer E.** Draw a vertical line through the point where the circles touch: it will bisect the square. Thus the horizontal distance from the centre of either circle to the nearest corner of the square is $1 - \frac{1}{2}t$, and the vertical distance is $1 - t$. Since the radius is 1, it follows by Pythagoras' theorem that $(1-\frac{1}{2}t)^2 + (1-t)^2 = 1$, so $1 - t + \frac{1}{4}t^2 + 1 - 2t + t^2 = 1$, giving $1 - 3t + \frac{5}{4}t^2 = 0$. This factorises as $(1-\frac{5}{2}t)(1-\frac{1}{2}t) = 0$, so $t = \frac{2}{5}$ or $t = 2$. The last answer is impossible, since from the diagram $t < 1$.

20. **Answer C.** For convenience, suppose the length of AB is $6p$ and the length of BC is $6q$. If R is the midpoint of EF and N is the midpoint of BG, then $RN = \frac{2}{3}BC = 4q$ and $AN = 5p$. By similar triangles, $\frac{HI}{AI} = \frac{RN}{AN}$, so $HI = \frac{RN}{AN}AI = \frac{4q}{5p}2p = \frac{8}{5}q$. The area of triangle AIH is therefore $\frac{1}{2}(2p)(\frac{8}{5}q) = \frac{8}{5}pq$. The ratio of the areas of AIH and $ABCD$ is $\frac{8}{5}pq : 36pq = 8 : 180 = 2 : 45$. [Notice that p and q cancel out at the end, so they are actually unnecessary.]