THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior First Round 2009
Solutions

1. **Answer A.** "Per cent" means "out of 100", so $6.28\% = \dfrac{6.28}{100} = 0.0628$. (Remember that when dividing by 100, the decimal moves two places to the left.)

2. **Answer E.** One revolution is $360°$, so four and a half revolutions is $4.5 \times 360° = 1620°$.

3. **Answer B.** Since the volume of 4000 drops is 330 mℓ, it follows that 12 000 drops have a volume of 990 mℓ, which is approximately one litre. The number of drops in a day is equal to the number of seconds in a day, which is $60 \times 60 \times 24$. The number of litres wasted in a day is therefore approximately $\dfrac{60 \times 60 \times 24}{12000} = 7.2 \approx 7$.

4. **Answer B.** If the mass of a red ball is r g, then the mass of three yellow balls and four red balls is $(3 \times 80 + 4r)$ g, which is equal to 420 g. Thus $240 + 4r = 420$, so $4r = 420 - 240 = 180$, and therefore $r = 180 \div 4 = 45$.

5. **Answer B.** The percentage change is given by the formula $\left(\dfrac{\text{New value} - \text{Original value}}{\text{Original value}}\right) \times 100$. In this case the new price is approximately R64 and the original price is approximately R80, so the percentage change is approximately $\left(\dfrac{64-80}{80}\right) \times 100 = -20$. The negative sign shows that the price has been reduced, and the price reduction is approximately 20%.

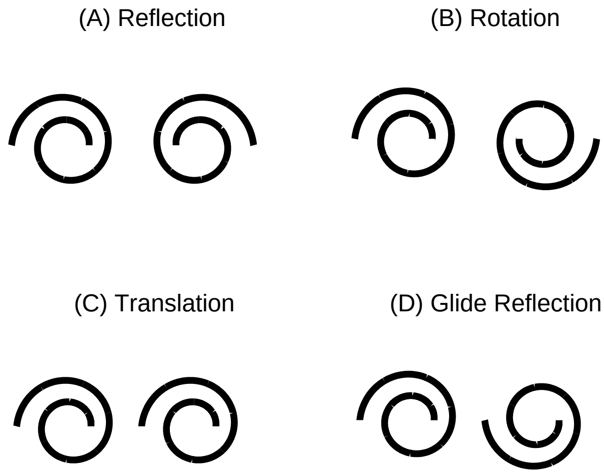
6. **Answer A.** From A to M the x-coordinate increases from 15 to 33, an increase of 18. Similarly the y-coordinate increases by 22, from 15 to 37. Since M is the centre of the rectangle, the co-ordinates will change by the same amounts when moving from M to C. Thus the coordinates of C are $(33 + 18\,; 37 + 22) = (51; 59)$. Alternatively, from A to C the coordinates will change by twice as much as from A to M, so C has coordinates $(15 + 36\,; 15 + 44) = (51; 59)$, as before.

7. **Answer C.** In ascending order, the guesses are 29, 31, 33, 35, 37. It is clear that the middle number differs from the outer two numbers by 4, and from the other two numbers by 2, as required.

8. **Answer A.** Since the pyramid has the same base as the cube, but tapers to a point while the cube has uniform cross-section, the pyramid obviously fits inside the cube, so it must have smaller volume. (Remember that all solids have the same height s.) Similarly, the cone has smaller volume than the cylinder. The sphere also fits inside the cylinder (just touching at the equator and the poles), so the sphere also has smaller volume than the cylinder. Finally, the circular base of the cylinder fits inside the square base of the cube, so the whole cylinder fits inside the cube. Thus the cube has the largest volume of all five solids.

9. **Answer E.** Since the tank is originally two-thirds full, the depth of the water is originally $\frac{2}{3} \times 60 = 40$ cm, so the volume of water is $50 \times 30 \times 40$ cm^3. When the rock is put in the tank, the depth increases to 42 cm, so the volume of water plus rock is $50 \times 30 \times 42$ cm^3. The volume of the rock is therefore $50 \times 30 \times 2$ cm$^3 = 3000$ cm^3. [This is an example of Archimedes' Principle, which says that when a solid is immersed in a liquid, the volume of the solid is equal to the volume of liquid displaced.]

10. **Answer B.**



One way to understand transformations is to trace one figure on a piece of transparent paper. Place the paper so that the tracing exactly covers the left-hand figure of each pair and then decide how to move the paper so as to make the tracing cover the right-hand figure. For the reflection (A) the paper must be turned over about a vertical axis. For the rotation (B) the paper must be rotated about a fixed point between the two figures. For the translation (C) the paper needs only to be shifted across. For the glide reflection (D) the paper must be turned over about a horizontal axis and also shifted sideways. [Transformations (B) and (C), in which the paper stays the same way up, are called *even* transformations, while (A) and (D), in which the paper is turned upside-down, are called *odd* transformations. Try and work out the four possible outcomes when an even or odd transformation is followed by an even or odd transformation. Does the result look familiar?]

[The *symmetries* of a figure are the transformations that leave the figure unchanged. Every figure is unchanged by the identity transformation, which leaves all points in the same place, so the identity is a symmetry of very figure. For example, if you take just one of the spiral figures above, then it has no symmetries except the identity, because there is no way you can make the tracing fit over the original spiral except by leaving the tracing paper in the same place. It is interesting to try to find the symmetries of the capital letters A to Z. The letter A is unchanged by a reflection in a vertical line, the letter B is unchanged by reflection in a horizontal line, the letter N is unchanged by a rotation. Now you can try the other letters for yourself. Some have more than one symmetry (besides the identity) and some have only the identity.

Another interesting example is the sine graph $y = \sin\theta$, where θ is measured in degrees. (You must allow θ to make infinitely many revolutions, both positive and negative, and not restrict the graph to $0 \leq \theta \leq 360$.) Here are some examples of the symmetries of the sine graph: (1) rotation about the origin, (2) translation by 360 parallel to the θ-axis, (3) reflection in the vertical line $\theta = 90$, (4) glide reflection made up of reflection in the horizontal line $y = 0$ (the θ-axis) followed by translation by 180 parallel to the θ-axis. You can find many more symmetries of the sine graph, and you can also try other graphs, trigonometric or otherwise.]

11. **Answer B.** We first need to factorize 2009. It is obviously not divisible by 2 or by 5, since the last digit isn't divisible by 2 or by 5. It is also not divisible by 3 since the sum of the digits isn't divisible by 3. The next prime to try is 7, and we see that $2009 = 7 \times 287$. It is now easy to see that $287 = 7 \times 41$, so $2009 = 7^2 \times 41$. In a perfect square, all prime factors have even powers, so we can multiply 2009 by 41 to obtain the smallest perfect square $7^2 \times 41^2$.

12. **Answer D.** Consider the following truth table of the four statements, depending on who has the key:

	Ann	Ben	Cal	Don
Ann has the key	False	False	True	False
Ben has the key	True	False	False	True
Cal has the key	False	True	False	True
Don has the key	True	True	True	False
No-one has the key	True	True	True	True

The only case in which there is exactly one false statement is if Don has the key. Alternatively, exclude each statement in turn, assume that the remaining three statements are true, and then check that the excluded statement is false. The only time this occurs is if Don's statement is false. [Each statement consists of two simpler statements, connected by "and". The only way the full statement can be true is when both simpler statements are true. Only one being true is not enough.]

13. **Answer D.** The probability that the first of the five digits is 1 is $\frac{1}{5}$. Then the probability that the last of the remaining four digits is 5 is $\frac{1}{4}$. Thus the probability of both occurring together is $\frac{1}{5} \times \frac{1}{4} = \frac{1}{20}$.
Alternatively, there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ numbers that use the five digits once each. Among these, once the first and last digits have been fixed, there are $3 \times 2 \times 1 = 6$ rearrangements of the middle three digits. Thus the probability is $\frac{6}{120} = \frac{1}{20}$.

14. **Answer D.** Let x be the area of the overlapping portion, let y be the area of the remainder of the larger square, and z the area of the remainder of the smaller square. Then $x + y = 6^2 = 36$ and $x + z = 4^2 = 16$, so $y - z = 36 - 16 = 20$.
[Note that the value of z does not affect the answer, so you could get the answer by assuming the smaller square is completely inside or completely outside the larger one.]

15. **Answer D.** Let $x = \sqrt{8+3\sqrt{7}} - \sqrt{8-3\sqrt{7}}$. Then clearly $x > 0$, and

$$x^2 = (8+3\sqrt{7}) - 2\sqrt{(8+3\sqrt{7})(8-3\sqrt{7})} + (8-3\sqrt{7}) = 16 - 2\sqrt{(8+3\sqrt{7})(8-3\sqrt{7})}.$$

Next $(8+3\sqrt{7})(8-3\sqrt{7}) = 8^2 - 3^2 \times 7 = 64 - 63 = 1$, so $x^2 = 16 - 2\sqrt{1} = 14$, and therefore $x = \sqrt{14}$.

16. **Answer B.** Join AM; then $\triangle ADM = 4\triangle PDM$. (The triangles have the same height, so their areas are proportional to their bases.) By subtraction, we see that $\triangle APM = 3\triangle PDM$. Next, $\triangle AMB = \triangle AMD$ by the same argument, since the bases MB and MD are equal. Thus $\triangle AMB = 4\triangle PDM$. Combining triangles APM and AMB we see that quadrilateral $ABMP = 7\triangle PDM$, so $\triangle PDM = 35 \div 7 = 5$.

17. **Answer A.** Using cm as units, let the length, width and height be a, b, c. Then $4a+4b+4c = 20$, so $a+b+c = 5$. If the diagonal of the face with sides a and b is d, then by Pythagoras' theorem $d^2 = a^2 + b^2$, and by using Pythagoras again we see that $d^2 + c^2 = 4^2$, since the corner-to-corner diagonal is 4. Thus $a^2 + b^2 + c^2 = 16$. The total surface area S is given by $S = 2ab + 2bc + 2ca$. Now

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca,$$

giving $5^2 = 16 + S$, so $S = 9$.

18. **Answer E.** Suppose the circles have radii R and r. The centres of the circles and the point where they touch divide the diagonal into four line segments with lengths $R\sqrt{2}$, R, $,r$, $r\sqrt{2}$. Since the diagonal has length $\sqrt{2}$, it follows that $(R+r)(\sqrt{2}+1) = \sqrt{2}$, so $R + r = \dfrac{\sqrt{2}}{\sqrt{2}+1} = 2 - \sqrt{2}$. (To rationalize the denominator, multiply top and bottom by $\sqrt{2} - 1$.)

19. **Answer D.** If two or more positive integers have a given sum, then the product will in general be greatest when the numbers are as near equal as possible. (For example, $10 \times 10 > 9 \times 11 > 8 \times 12$.) The problem is to find out how many numbers to take. With four 5s we get $5^4 = 625$ (greater than $10^2 = 100$), and with five 4s we get $4^5 = 1024$, which is even greater. If we use only 3s we cannot get a sum of 20 exactly, but we could try five 3s and a 5, giving $3^5 \times 5 = 1215$, or six 3s and a 2, giving $3^6 \times 2 = 1458$, which is the greatest so far. Now continue and try ten 2s. The product is $2^{10} = 1024$ again, which is less than 1458, so 1458 is the correct answer. [If we were allowed to use real numbers instead of integers, then the maximum value would be $e^{20/e} \approx 1568.05$, obtained by repeating the number $e \approx 2.718$. With integers, as shown above, the largest product uses six 3s and a 2, whose average is $2\frac{6}{7} \approx 2.857$.]

20. **Answer D.** Let x be the sum of the second series. If we subtract the second series from the first, then the odd terms disappear and we see that

$$\frac{\pi^2}{6} - x = 2(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots).$$

We can now take out a common factor 2^2 from all the denominators to obtain

$$\frac{\pi^2}{6} - x = \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots) = \frac{\pi^2}{12}.$$

Thus $x = \frac{\pi^2}{6} - \frac{\pi^2}{12} = \frac{\pi^2}{12}$.
[Euler was the first person to evaluate the sum of the first series. Now see if you can evaluate the sum of the odd terms in that series.]