The South African Mathematical Olympiad

Junior Third Round 2011

Solutions

1. One micro century is a millionth of a century, which can be approximated by

$$\begin{aligned}\frac{100\text{ years}}{1000000} &\approx \frac{100\times 365}{1000000}\text{ days} \quad \text{(ignoring leap years)}\\ &= \frac{365\times 24\times 60}{10000}\text{ minutes}\\ &= \frac{73\times 12\times 6}{100}\text{ minutes}\\ &\approx 53\text{ minutes.}\end{aligned}$$

   Alternatively, one could approximate as follows:

$$\begin{aligned}\frac{100\text{ years}}{1000000} &\approx \frac{100\times 365}{1000000}\text{ days} \quad \text{(ignoring leap years)}\\ &= \frac{365\times 24}{10000}\text{ hours}\\ &= \frac{8760}{10000}\text{ hours}\\ &= 0.876\text{ hours}\\ &\approx 0.875\times 60\text{ minutes}\\ &= 8\tfrac{3}{4}\times 6\text{ minutes}\\ &= 52\tfrac{1}{2}\text{ minutes.}\end{aligned}$$

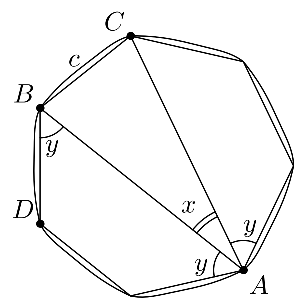
2. Suppose that $N^2 - 200 = x^2$. Then $N^2 - x^2 = (N-x)(N+x) = 200$. Note that $N-x$ and $N+x$ are either both odd, or both even. Since their product is equal to 200, which is even, we must have that both $N-x$ and $N+x$ are even. We can thus write $\left(\frac{N-x}{2}\right)\left(\frac{N+x}{2}\right) = \frac{200}{2\times 2} = 50 = 2\times 5\times 5$.

   Since $\frac{N-x}{2}$ and $\frac{N+x}{2}$ are integers with $\frac{N-x}{2} < \frac{N+x}{2}$, it follows that the only possible values for $\frac{N-x}{2}$ is 1, 2 or 5, with corresponding values of $\frac{N+x}{2}$ equal to 50, 25 and 10.

   Solving these simultaneous equations, we get the three solutions $(N, x) = (51, 49)$, $(27, 23)$ and $(15, 5)$.

3. Suppose that the triangle has side lengths $n-1$, n and $n+1$. Since the hypotenuse is the longest side, we must have $(n-1)^2 + n^2 = (n+1)^2$, which simplifis to $n^2 - 4n = n(n-4) = 0$, which has solutions $n = 0$ and $n = 4$. We may disregard the solution $n = 0$ since it doesn't result in a triangle, so $n = 4$ is the only solution, corresponding to the triangle with side lengths 3, 4 and 5.

4. Since the coin is symmetric, its circumference will be $7c$, where c is the length of the arc of the circle with centre A and radius Q, as shown. To compute the length of c, we need to find the angle x.



The interior angles of a regular heptagon is equal to $\frac{180\times 5}{7} = \frac{900}{7}$ degrees. By symmetry, all the angles y are equal to $\frac{1}{2}\left(\frac{900}{7} - x\right)$. Hence we have

$$\begin{aligned}\frac{900^\circ}{7} &= \angle DBC \\ &= \angle DBA + \angle CBA \\ &= \frac{1}{2}\left(\frac{900^\circ}{7} - x\right) + \frac{1}{2}(180^\circ - x).\end{aligned}$$

Solving this for x yields the solution $x = \frac{180}{7}$ degrees. The length of c is thus equal to

$$2\pi Q \times \frac{x}{360^\circ} = \frac{\pi Q}{7},$$

and the circumference of the coin is thus equal to $7c = \pi Q$.

5. (a) Cut on the line between the centre of the circle and the centre of the rectangle. This line divide both the area of the rectangle and the area of the circle in half, hence it will divide the shaded area in half.

   (b) The procedure in part (a) works for any circle inside the rectangle, except if the centre of the circle is also the centre of the rectangle. In that case, however, *any* line through the centre of the circle will work, since it still divides both the area of the circle and the area of the rectangle in half.

6. The potatoes originally contained 1 kg $= 1\% \times 100$ kg "potato matter". After some water has evaporated, this 1 kg matter now forms 2% of the total mass. So the total mass must now be equal to $\frac{100\%}{2\%} \times 1$ kg $= 50$ kg.

7. From the given equation it is clear that $ABCDE = \frac{EDCBA}{4} < \frac{100000}{4} = 25000$, so $A \leq 2$. However, since $EDCBA$ is a multible of 4, A must be even, so $A = 2$.

   This now means that E equals 8 or 9, since $ABCDE \times 4 > 20000 \times 4 = 80000$. But the unit digit of $4 \times E$ must equal $A = 2$, so $E = 8$.

From the multiplication it now follows that B is the units digit of $4 \times D + 3$ (the 3 is carried over from $8 \times 4 = 32$), which is odd. On the other hand, $2BCD8 \times 4 = 8DCB2 < 90000$, so $\frac{2BCD8}{4} < \frac{90000}{4} = 22500$. This shows that $B \leq 2$, and B odd now implies that $B = 1$.

This now means that $D \times 4 + 3$ must end in a 1, so $D \times 4$ ends in 8, so $D = 2$ or $D = 7$. However, $D \neq 2$ (since $A = 2$), so $D = 7$.

Finally, $4 \times C + 3$ ends in C, which is the same as saying that $3 \times C + 3$ ends in 0. The only possibility is $C = 9$.

8. (a) One solution is $2 \times 6 + 8 + 4 = 24$.
   (b) $8 \div (3 - (8 \div 3)) = 24$.

9. Let the uphill distance from A to B be a km, the level distance be b km and the downhill distance be c km. Then, using the formula time $= \frac{\text{distance}}{\text{speed}}$, the trip from A to B takes $\frac{a}{56} + \frac{b}{63} + \frac{c}{72}$ hours, while the return trip takes $\frac{a}{72} + \frac{b}{63} + \frac{c}{56}$ hours (the uphill and downhill distances are interchanged for the return trip). Thus we have

$$\begin{aligned} \frac{a}{56} + \frac{b}{63} + \frac{c}{72} &= 4 \\ \frac{a}{72} + \frac{b}{63} + \frac{c}{56} &= 4\tfrac{2}{3}. \end{aligned}$$

   Adding these equations together yields

$$8\tfrac{2}{3} = \left(\frac{1}{56} + \frac{1}{72}\right) a + \left(\frac{2}{63}\right) b + \left(\frac{1}{72} + \frac{1}{56}\right) c = \frac{2}{63}(a + b + c).$$

   This gives the distance between A and B as $a + b + c = \frac{63}{2} \times \frac{26}{3} = 273$ km.

10. Suppose there are N people in the queue, with w women. Suppose that Saskia joins the queue such that there are n people in front of her, of which m are men. We now calculate how many women are behind her.

    Since there are n people in front of her, of which m are men, there are $n - m$ women in front of her. Since there are w women in total, there must be $w - (n - m) = w + m - n$ women behind her. Saskia would like this number to equal m, the number of men in front of her. Equating the two numbers gives $w + m - n = m$ which simplifies to $n = w$.

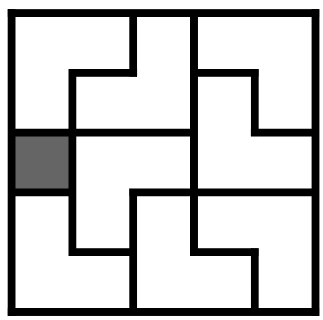
    This means that if Saskia joins the queue in the position where the number of people in front of her is equal to the total number of women in the queue, then the number of men in front of her will equal the number of women behind her, as required.

    Alternative solution

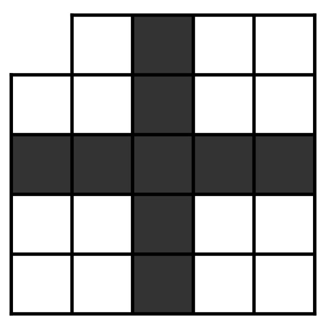
    At any point in time, let M be the number of men in front of Saskia, and let W be the number of women behind Saskia. If Saskia joins the back of the queue, we initially have $W = 0$ and $M \geq 0$. If $M = 0$ (i.e. there are no men in the queue), then we are done. If not, Saskia can skip one place ahead. If Saskia skipped over a woman, then W increases by 1, while M stays the same, so the number $M - W$ decreases by one. If Saskia skipped over a man, then W stays the same, while M decreases by one, so the number $M - W$ decreases by one. In other words, every time Saskia moves one place forward in the queue, the number $M - W$ decreases by one. initially this number is positive (since initially we have $M > W = 0$), and by the time Saskia reaches the front of the queue, $M - W \leq 0$ since $M = 0$ at that point. Since $M - W$ decreases by exactly 1 at each step, starts out being positive and ends up being non-negative, there must be a point where it reaches 0, i.e. $M = W$, which means that at that point the number of men ahead of Saskia equals the number of women behind her.

11. (a) Any tiling by L-shapes must cover a multiple of 3 tiles, and $5 \times 5 = 25$ is not a multple of 3, so it is impossible.

    (b) Yes, it is possible, for example:



    (c) Colour the centre cross of the board black, as shown. Whenever a rectangle is placed on the board, it will either cover 2 white squares and one black square, or three black squares. This means that the total number of white squares covered is even. However, there are 15 white squares on the board, which is odd. This shows that such a tiling is impossible.



12. The answer depends on the starting position. If the two coins are not next to each other, then the first player simply moves the leftmost coin next to the rightmost coin. This forces player 2 to now move the rightmost coin, which creates a gap between the two coins. Player one can now always move the leftmost coin next to the rightmost coin. Every time the second player moves, it enables the first player to move, so eventually the second player must lose.

However, if the coins start out next to each other, the roles are reversed; the first player is now forced to move the rightmost coin to create a gap between the two coins, which, using the above strategy, shows that the second player now wins.

13. Let the number of questions Susan answered correctly be c, the number she answered incorrectly be i and the number of unanswered questions be u. Note that $c + i + w = 26$.

   (a) If she answered all the questions, $u = 0$, so $c + i = 26$ and

   $$0 = 8c - 5i = 8c - 5(26 - c) = 13c - 130,$$

   which gives the unique solution $c = 10$.

   (b) If some questions were unanswered, $0 < u \leq 26$, and $8c - 5i + 2u = 0$. Then

   $$0 = 8c - 5i + 2u = 8c - 5(26 - c - u) + 2u = 13c - 130 + 7u.$$

   This shows that u must be a multiple of 13, and $u = 26$ is not possible since this gives a total score of $26 \times 2 = 52$, so we must have $u = 13$. Therefore

   $$0 = 13c - 130 + 7 \times 13 \implies 0 = c - 10 + 7,$$

   which gives $c = 3$.

14. The pills, swallowed in order, are

   $$2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 3,\ 7,\ 11.$$

   (a) The first black pill in the above list is pill 7, swallowed by prisoner number 8.

   (b) The only numbers not appearing in the above list are 1, 5 and 9. If no prisoner is to die, these three pills must be coloured black.

15. Let the greatest integer dividing all the numbers equal k. Then k must divide $1(1+1)^2(1+2)^3(1+3)^4 = 2^{10}3^3$, so k can only have 2 and 3 as prime factors. Let $k = 2^a 3^b$.

   Now, k must divide $4(4+1)^2(4+2)^3(4+3)^4 = 2^5 3^3 5^2 7^4$, so $a \leq 5$. Also, k must divide $2(2+1)^2(2+2)^3(2+3)^4 = 2^7 3^2 5^4$, so $b \leq 2$. We now show that $k = 2^5 \times 3^2$.

   For every n, either n and $n+2$ are even, and one of them is a multiple of 4, or $n+1$ and $n+3$ are even, with one being a multiple of 4. In either case, 2^5 is a factor of $n(n+1)^2(n+2)^3(n+3)^4$.

   Similarly, for every n, exactly one of $n+1$, $n+2$ and $n+3$ is a multiple of 3. Hence $n(n+1)^2(n+2)^3(n+3)^4$ is always divisible by 3^2. Hence $k = 2^5 \times 3^2$.