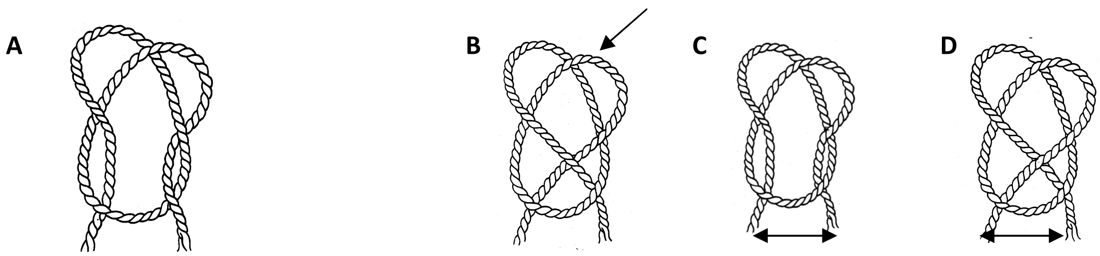
SAMO 2009 Junior Round 3 Solutions

Question 1

Grasp the two loose ends of each rope firmly in your mind. Then imagine yourself pulling them until you have a straight piece of rope – either with a knot or without one.
Which of these four ropes will give you a knot?

A will form a knot.
In rope B, the top right hand loop can be pulled down through the top left hand loop, leaving no knot.
The bottom two strands in ropes C and D can be crossed to form rope B, which can then be untangled.



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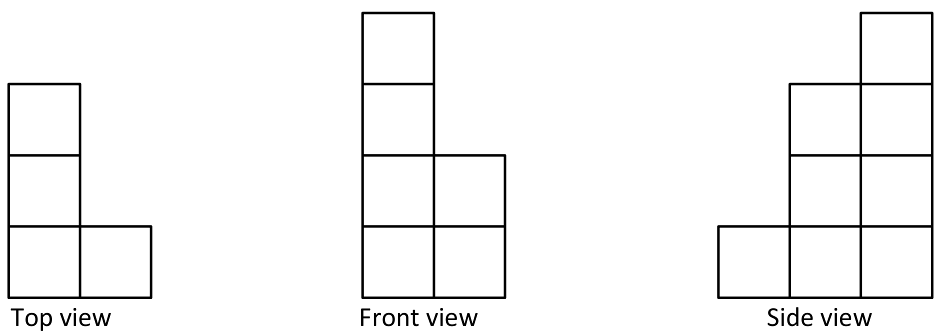
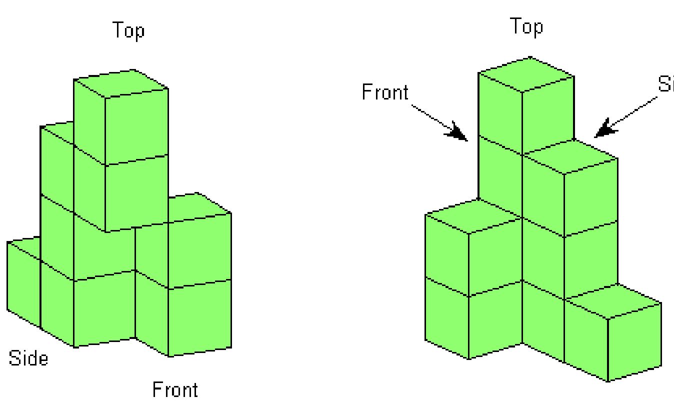
Question 2

A structure is built with identical cubes. The top view, the front view and the side view are shown below.
What is the least number of cubes required to build this structure?



The structure can be built using 8 cubes, shown in the picture below.



8 cubes are visible in the side view, so this must be the minimum number of cubes.

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If the answer was given as 10, reasoning that the two cubes at the top needed to be supported, full marks were awarded.

Question 3

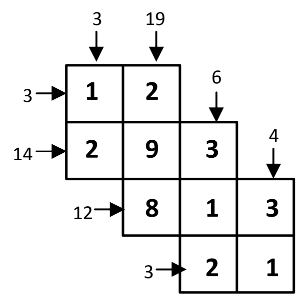
The digits of a two-digit number AB are reversed to give the number BA. These two numbers are added. For what values of A and B will the sum be a square number?

AB + BA = (10A + B) + (10B + A) = 11A + 11B = 11(A+B)
A+B must be a multiple of 11, so
(A, B) = (9,2) (2,9) (8,3) (3,8) (7,4) (4,7) (6,5) (5,6)

6

Question 4

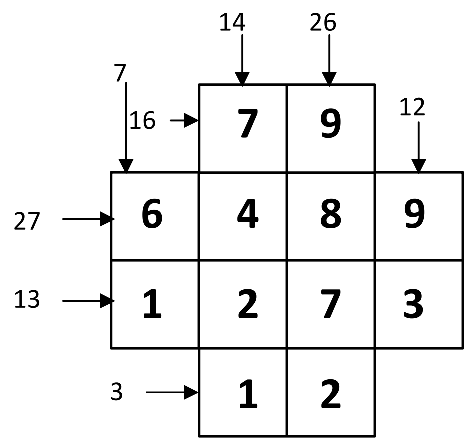
Example



Much like crossword puzzles, there are also Cross Number puzzles in which numbers from 1 to 9 need to be filled into the blocks in such a way that each vertical column and horizontal row adds up to the number shown above that column or to the left of that row. A number may not be repeated in any row or column.

An example is given.

Now complete this Cross Number puzzle.



2 numbers summing to 3 (bottom row) must be 2 + 1. The only way that 4 numbers including either a 1 or 2 can sum to 26 is 9 + 8 + 7 + 2, so the bottom row is 1 2.
2 different numbers summing to 16 (top row) must be 9 + 7. If the 9 lies in the column summing to 14 then the other 3 numbers must sum to 5, which is impossible. The top row is therefore 7 9.
4 numbers summing to 14 including a 1 and 7, must be 7 + 4 + 2 + 1
4 numbers making 27 including either a 4 or 2, must be 9 + 8 + 6 + 4, so the 14 column is 7 4 2 1, and the 26 column is 9 8 7 2.
The 27 row must now be 6 4 8 9, making the 13 row 1 2 7 3.

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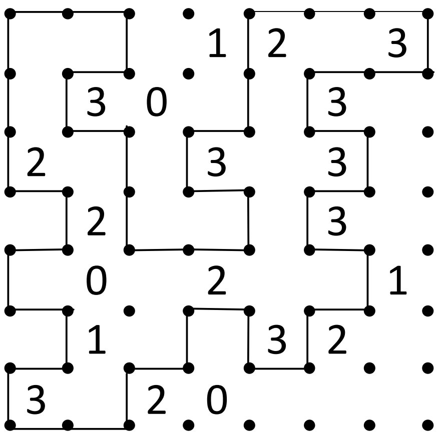
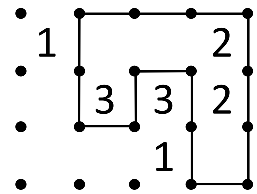
Question 5

Connect adjacent dots with vertical or horizontal lines so that a single closed loop is formed with no crossings or branches. Each number indicates how many lines surround it, while empty cells may be surrounded by any number of lines.

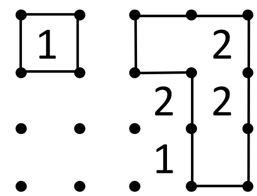
One correct and three incorrect examples are given.



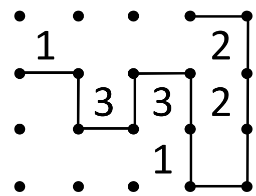
Correct Example



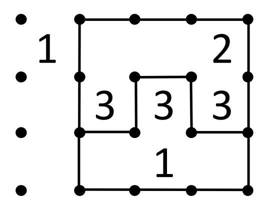
Incorrect Examples



(Two closed loops)



(Not a closed loop)



(Has branches)

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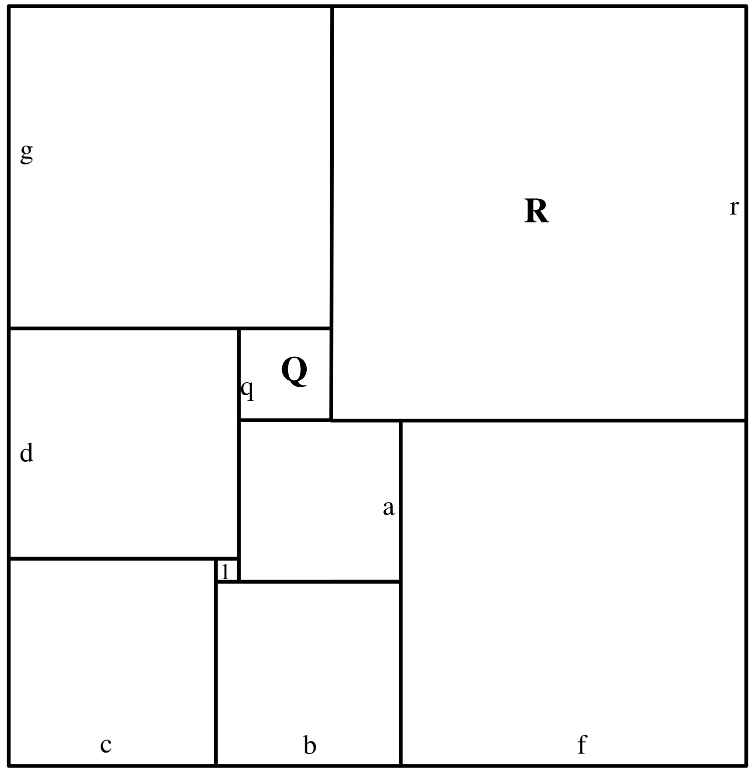
Question 6

Nine squares are arranged to form a rectangle as shown in the diagram.

Square P has an area of 1.

a) Find the area of square Q.

b) Prove that the area of square R is 324 times that of square P.



a) $q + a = d + 1$
$= c + 2$
$= b + 3$
$= a + 4$

So $q = 4$

The area of Q is q^2 = 16

b) $r = g + q$
$= d + 2q$
$= d + 8$
$= a + 11$

and $r = a + f - q$
$= 2a + b - q$
$= 3a + 1 - q$
$= 3a - 3,$

so $a + 11 = 3a - 3$. a is then 7, and $r = 18$.

The area of R is r^2 = 324

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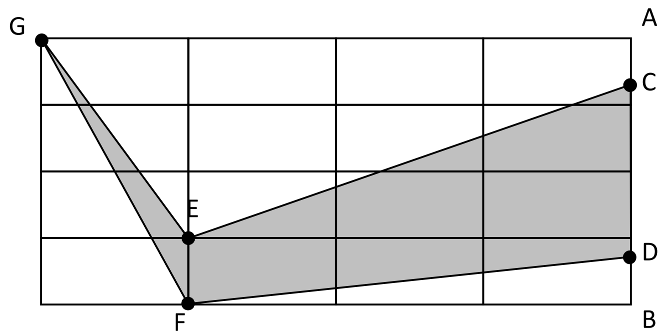
Question 7

In the figure the small rectangles are identical and each has an area of 8 cm^2.

C and D are points on the line segment AB as shown.

If $CD = \frac{2}{3}AB$**, find the shaded area in cm^2.**



Let the shorter side of the rectangles be a, and the longer side be b. Then $ab = 8$.

Total shaded area = Area FGE + Area EDF + Area CDE

Area FGE $= \frac{1}{2} \times base \times height = \frac{1}{2} \times a \times b$

Area EDF $= \frac{1}{2} \times a \times 3b$

Area CDE $= \frac{1}{2} \times \left(\frac{2}{3} AB\right) \times 3b = \frac{1}{2} \times \frac{2}{3} \times 4a \times 3b$

Total $= \frac{1}{2}ab + \frac{3}{2}ab + 4ab = 6ab = 48$ cm^2.

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Question 8

On a distant planet, railway tracks are built using one solid railway bar. A railway is built between two towns 20 km apart on a big flat section of the planet. Unfortunately the bar was made one metre too long and the constructor decided to lift it in the middle to try to make the ends fit.

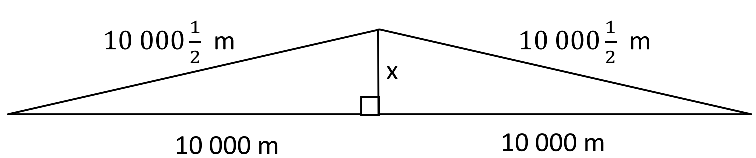
Approximately how high does he have to lift it in the middle?
Is it 1 cm, 10 cm, 1 m, 10 m, 100 m or 1 km?

a) **Guess one of the above, without doing any calculations.** **(1)**

b) **Calculate the answer and comment on how it compares with your guess.** **(5)**

20 km + 1 m = 20 001 m

Each half of the lifted bar has length $10\,000\frac{1}{2}$ m



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$$x^2 + 10\,000^2 = (10\,000 + \frac{1}{2})^2$$

$$x = \sqrt{(10\,000 + \tfrac{1}{2})^2 - 10\,000^2} = \sqrt{10\,000^2 + 2 \times 10\,000 \times \tfrac{1}{2} + (\tfrac{1}{2})^2 - 10\,000^2} = \sqrt{10\,000 + \tfrac{1}{4}} \cong \sqrt{10\,000} = 100 \text{ m}$$

Question 9

Two candles of the same height are lit at the same time. The first candle is completely burnt up in 3 hours while the second candle is completely burnt up in 4 hours. At what point in time is the height of the second candle equal to twice that of the first candle?

Let the candles have height 1. The first candle burns up in 3 hours, so it burns at a rate of $\frac{1}{3}$ per hour. After time t, it will have burnt up $\frac{t}{3}$, so its height will be $1-\frac{t}{3}$

The second candle burns at a rate of $\frac{1}{4}$ per hour. After time t, it will have burnt up $\frac{t}{4}$, so its height will be $1-\frac{t}{4}$.

When the height of the second candle is twice that of the first,

$1-\frac{t}{4}=2(1-\frac{t}{3})$

Solving for t gives $t=\frac{12}{5}$ hours, or 144 minutes.

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Question 10

Nick and John play the following game. They put 100 pebbles on the table. During any move, a player takes at least one and not more than eight pebbles. Nick makes the first move, then John makes his move, then Nick makes a move again and so on. The player who takes the last pebble is the winner of the game.

a) What strategy can you offer Nick to win the game?

b) Can you offer John, as the second player, such a strategy?
(Give reasons for your answers)

a) Nick should take 1 pebble in his first move, leaving 99 pebbles behind. For the next moves, whenever John takes x pebbles (x between 1 and 8), Nick should take 9-x pebbles (9-x will also be between 1 and 8). This will always leave a multiple of 9 pebbles for John's turn. Eventually, there will be 9 pebbles at John's turn. He cannot take all 9, but however many he takes, he will leave behind 8 or fewer pebbles. Nick can then take all the remaining pebbles and win.

b) John cannot have a winning strategy. Nick's winning strategy from part (a) means that whatever move John makes, Nick can choose a move to win.

8

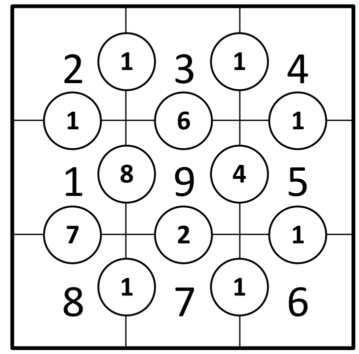
Question 11

The numbers 1 to 9 must be placed in the squares on the grid. The numbers in the circles are the positive differences between the numbers in adjacent squares.

S is the sum of the numbers in the circles.

An example is given.

Example

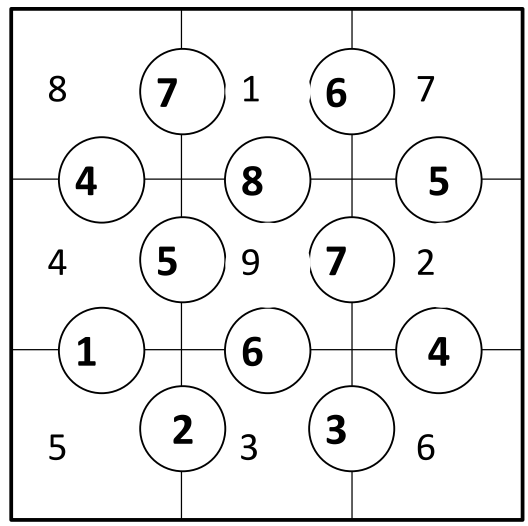


S = 34

a) **Show how to arrange the numbers 1 to 9 in the squares so that S is a maximum.**

b) **Explain clearly why no other arrangement could give a larger total than yours.**

a) The maximum value is $S = 58$
An example giving S =58 is shown.



b) 9 or 1 is placed in the middle.
The lowest numbers/or highest numbers are placed adjacent to it. The remaining numbers can be placed anywhere in th remaining squares as it won't make a difference and will add up to the same total.

Therefore,
if 9 is in the middle, 5, 6, 7 and 8 should be in the corners or if 1 is in the middle, 2, 3, 7 and 5 should be in the corners.

Question 12

Place algebraic operations $+$; $-$; $\div$ or $\times$ between the numbers 1 to 9 in that order so that the total equals 100. You may also freely use brackets before or after any of the digits in the expression and numbers may be placed together, such as 123 and 67 (see example).

Two examples are given below:

i) $123+45-67+8-9=100$

ii) $1+[(2+3)\times4\times5]-[(6-7)\times(8-9)]=100$

Four solutions will be awarded 2 marks each. Any other solution will get a bonus of 1 mark each to a maximum of 3 bonus marks.

A computer search gives over 10 000 possible solutions.
Every answer should be checked!
Examples of correct answers:

$(12+3)\times4+5\times6-7+8+9$

$123-4-5-6-7+8-9$

$(1+234)\div5+6+7\times8-9$

$1+(2+3+4-5+6-7+8)\times9$

etc.

No marks for: $123+45-67+8-9=100$

$1+[(2+3)\times4\times5]-[(6-7)\times(8-9)]=100$

8	+	3 max

Question 13

(a) **Find three different positive integers, the sum of any two of which is a perfect square.** **(3)**

(b) **Find a general formula that will help generate other such triplets.** **(5)**

(a) For example: -2; 3 en 6 of -4; 5 en 20

(b) If the numbers are a, b, c, and $a+b = x$, $b+c = y$ and $c+a = z$, then $y = x-2a+z$.

For any integer a, choosing $z=1$ and $x = a^2$, which are perfect squares, means that $y = a^2-2a+1 = (a-1)^2$ is also a perfect square.

If a is not 1 or 0, then a, a^2-a, and $1-a$ are 3 non-zero integers with

$$a+(a^2-a) = a^2$$
$$(a^2-a)+(1-a) = (a-1)^2$$
$$(1-a)+a = 1^2.$$

3	+	5

Question 14

An age-old problem states the following:

A camel sits next to a pile of 3 600 bananas at the edge of a desert. He has to get as many bananas as possible, across this desert which is 1 000 km wide. He can only carry a maximum of 1 200 bananas at any one time. To survive he has to eat one banana for each kilometer he travels.

What is the maximum number of bananas that he can get to the other side of the desert?

The camel can get **840 bananas** to the other side of the desert.

Say the camel has n bananas that he wants to carry a certain distance. He can carry 1200 at a time, so he will have to take $\left\lceil \frac{n}{1200} \right\rceil$ trips. To return each time to fetch the next batch, he will need $\left\lceil \frac{n}{1200} \right\rceil - 1$ return trips. Therefore he will need to eat $2 \times \left\lceil \frac{n}{1200} \right\rceil - 1$ bananas in total for every kilometer along the way.

$\left\lceil \frac{3600}{1200} \right\rceil = 3$, and $2 \times \left\lceil \frac{3600}{1200} \right\rceil - 1 = 5$, so at first he will need to take 3 trips, and eat 5 bananas in total for every kilometer. However, after 240 km, he will have eaten $5 \times 240 = 1200$ bananas, so he will have 2400 bananas left. Now $\left\lceil \frac{2400}{1200} \right\rceil = 2$, and $2 \times \left\lceil \frac{2400}{1200} \right\rceil - 1 = 3$, so he will need to take 2 trips, and eat 3 bananas in total for every kilometer.

After another 400 km, he will have eaten another $3 \times 400 = 1200$ bananas, and have 1200 bananas left. Now he can carry all the bananas at once, so he only needs to take 1 trip, and eat 1 banana per kilometer. He has $1000 - 240 - 400 = 360$ km to go, so at the end he will have $1200 - 360 = 840$ bananas left.

8

OR

CAMEL ________________*x*______________________________*y*_________________Destination

If there are more than 2 400 bananas, three trips from the starting point will be necessary. Since there are 3600 bananas, the camel has to make three trips to some point x units away from the start point. Counting the back and forth travelling, gives a total of $5x$. Therefore

$3\,600 - 5x = 2\,400$, hence $x = 240$.

This means that the camel has to drop off the first load (of 1 200 bananas) 240 km from the start point. To complete the first full cycle trip, the camel eats 480 bananas, after the second, another 480 and the last load will cost 240 bananas. Therefore, at point x, we are left with 2 400 bananas.

Since there are more than 1 200 bananas at point x, two trips will be necessary to carry these to some point y units away from x. This is one full cycle trip and one one-way trip (total of three). Therefore

$2\,400 - 3y = 1\,200$. So $y = 400$

This means that the camel has to drop off the first load of (1 200 bananas) 400 km from point x. Therefore, moving bananas from point x to point y will cost us a total of 1 200 bananas. This leaves us with 1 200 bananas at point y. Note that point y is 360 km from the final destination. The remaining 1 200 bananas can now be carried across the remaining 360 km at a cost of 1 banana per km, leaving us with **840 bananas**.

8

Question 15

a) **Prove that** $3^{28} + 7^{51}$ **is not a prime number.** **(2)**

b) **Prove that** $2^{2009} + 5^{2010}$ **is not a prime number.** **(6)**

a) 3^x is always odd, and 7^x is always odd. Therefore $3^{28} + 7^{51}$ is a multiple of 2 (and it is greater than 2), so it cannot be prime.

2

b) Now find the remainders of 2^x when divided by 3.

+

$2^1 = 2$	remainder	2
$2^2 = 4$		1
$2^3 = 8$		2
$2^4 = 16$		1

If x is odd, 2^x has remainder 2, so 2^{2009} has remainder 2 when divided by 3. Since 2010 is even, 5^{2010} has remainder 1 when divided by 3. The two remainders add up to 3, so the total is divisible by 3.

$2^{2009} + 5^{2010}$ is divisible by 3, so it is not prime.

6