SAMO Junior Round 3 2007

Solutions by James Ridley

1. The sequence repeats in blocks ABCDEFGFEDCBA of length 13. Now 13 goes into 2007 a total of 154 times, leaving a remainder of 5. Thus before the 2007th letter there are 154 complete blocks, and the 2007th letter is the fifth letter in its block, so it is E.

2. Let angle ABF = angle AFB = x. Then angle EFD = x also (vertically opposite angles), so $b = c + x$ (exterior angle of triangle). Also $a + b + 2(180 - x) = 360$ (sum of the angles in quadrilateral CDFB), so $a + b = 2x.$ Eliminating x from these equations gives $a = b - 2c.$

3. (a) Ring 3 lies between circles of radius 3 and radius 4, so its area is $\pi(4^2 - 3^2) = 7\pi.$
(b) Ring 2007 lies between circles of radius 2007 and 2008, so its area is $\pi(2008^2 - 2007^2) = \pi(2008 + 2007)(2008 - 2007) = 4015\pi.$

4. The graph is continuous and increasing throughout. It is a straight line up to the level of the first shoulder, a steeper straight line to the next shoulder, then an even steeper straight line to the bottom of the conical section. From there it starts off with the same slope, but the slope decreases steadily, so the graph is curved with the concave side down. [The slope of the graph at any given height is inversely proportional to the cross-sectional area of the container at that height.]

5. If we write out the sum of the reciprocals of the factors $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{40} + \frac{1}{60} + \frac{1}{120}$, and bring it to a common denominator $\frac{120 + 60 + 40 + \cdots + 3 + 2 + 1}{120}$, then the numerator is the sum of the factors of 120 (in reverse order), so it is equal to 360. Thus the sum of reciprocals equals $360/120 = 3.$ [In general, if the sum of the factors of a natural number n is s, then the sum of the reciprocals of the factors of n is s/n.]

6. If the hypotenuse is y, then $12^2 = y^2 - x^2 = (y + x)(y - x),$ so the product of $y + x$ and $y - x$ is 144. Now $y + x$ and $y - x$ differ by $2x$, so they are either both even or both odd. Since their product is even, they cannot both be odd, so they are both even. Remembering that $y + x > y - x,$ we obtain the following four possibilities for two even numbers whose product is 144:

$y+x$	72	36	24	18
$y-x$	2	4	6	8
y	37	20	15	13
x	35	16	9	5

7. If the hypotenuse is y, then $y^2 = 8^2 + 15^2$, so $y = 17.$ The three right-angled triangles are all similar to one another, so $\frac{x}{8} = \frac{15}{17},$ giving $x = \frac{120}{17}.$

8. [Needs picture, viewed from the top of log A.] $(R - s)^2 + r^2 = R^2,$ so $R - s = \sqrt{R^2 - r^2}$, giving $s = R - \sqrt{R^2 - r^2}.$ [Viewed in two dimensions, as in the photograph, the curves of intersection are hyperbolas.]

9. The initial information we are given appears in the table below:

S	A	E	SA	SE	AE	SAE
0				9		5

Next A+SA+AE+SAE = 20 and SA+SAE = 12, so A+AE = 8 and SA = 12–5 = 7. Also E+SE+AE+SAE = 18, so E+AE=4. Finally, since the total number of players is 30, we have A+E+AE = 30–(0+7+9+5) = 9. It is now easy to see that A = 5, AE = 3, and E = 1.

10. (a) $\frac{2}{3}+\frac{5}{4}=\frac{7}{7}=1.$

(b) $\frac{b+a}{ab}=\frac{2}{a+b}$, so $(a+b)^2=2ab,$ giving $a^2+b^2=0,$ for which the only solution is $a=b=0,$ which is impossible, since a and b are in the denominators.

(c) $\frac{ad+bc}{bd}=\frac{a+c}{b+d}$, giving $abd+bcd+ad^2+b^2c=abd+bcd$, which simplifies to $ad^2+b^2c=0.$ There are many solutions with a, b, c, d all distinct and with b, d, and $b+d$ all non-zero. For example, $a=1,\ b=-1,\ c=-4,\ d=2$ gives the equation $\frac{1}{-1}+\frac{-4}{2}=\frac{1-4}{-1+2}.$

11. (a) $1+2=3$ (no brackets at the end), then $1+2(1+2)=7$ (one bracket), $1+2(1+2(1+2))=15$ (two brackets), and $1+2(1+2(1+2(1+2)))=31$ (three brackets).

(b) Each expression is one less than a power of 2 (the examples above are $2^2-1,$ $2^3-1,\ 2^4-1,$ and 2^5-1). Thus the expression with n brackets is equal to $2^{n+2}-1,$ and the expression with 2007 brackets is equal to $2^{2009}-1.$ [The general formula $2^{n+2}-1$ can be proved by noting that each expression is equal to 1 plus twice the previous expression.]

12. Move counter A to block 10 (End). If your opponent moves B or C to the End, then you can move the remaining counter to block 9, leaving your opponent with the last move. If he or she moves B or C to block 9, then you can move the remaining counter to the End, and again your opponent must make the last move.

13. (a) $65\times69=60\times74+5\times9=4440+45=4485.$

(b) Multiplying in the usual way gives $(10x+y)\times(10x+z)=100x^2+10x(y+z)+yz.$ Using the new method gives $10x\times(10x+y+z)+yz=100x^2+10x(y+z)+yz,$ which is the same.

14. With no jewels, you can score all multiples of 5, from 0 onwards. With one jewel, you can score all numbers with remainder 4 after division by 5, from 9 onwards. Now draw up a table:

Number of jewels:	0	1	2	3	4
Lowest possible score:	0	9	18	27	36
Remainder after division by 5:	0	4	3	2	1

Now all possible remainders have been covered, so the largest impossible score must be the largest number that is less than 36 and leaves remainder 1 after division by 5, which is 31.

15. If you bracket off the terms in pairs from the left, then there are 1003 brackets of the form $(n^3-(n-1)^3)$, followed by +1 at the end. Now $(n^3-(n-1)^3)=3n^2-3n+1,$ so each bracket leaves remainder 1 after division by 3. The whole expression therefore has the same remainder as $1003+1$ after division by 3. After dividing 1004 by 3, the remainder is 2. Thus the expression is not divisible by 3. [The remainder after dividing a number by 3 is the same as the remainder after dividing the sum of its digits by 3.]