SAMO, Solutions: Third round Junior, 2006

1. $10^6 = 2^6 \times 5^6$

$= 64 \times 15\,625$

2. Multiply by 1000 and divide by 8:

$$8\overline{)968\,880\,726\,456\,484\,032\,000}$$
$$121\,110\,090\,807\,060\,504\,000$$

3. By inspection:

1; 1; 1; 2; 5

1; 1; 1; 3; 3

1; 1; 2; 2; 2

OR

The numbers must be small (or else the product will be smaller than the sum), so suppose three of them are equal to 1, say 1; 1; 1; *x*; *y*

Then $3 + x + y = xy$

$\therefore xy - x - y = 3$

$\therefore (x-1)(y-1) = 4$

$\therefore x - 1 = 1$ and $y - 1 = 4$

or $x - 1 = 2$ and $y - 1 = 2$

This gives two solutions: 1; 1; 1; 2; 5 and 1; 1; 1; 3; 3

Next: Suppose two are equal to 1 and one is equal to 2:

Then $4 + x + y = 2xy$

$\therefore 4xy - 2x - 2y = 8$

$\therefore (2x-1)(2y-1) = 9$

$\therefore 2x - 1 = 1$ and $2y - 1 = 9$

or $2x - 1 = 3$ and $2y - 1 = 3$

This gives two solutions: 1; 1; 1; 2; 5 which we already have

and 1; 1; 2; 2; 2

4.

$\frac{1}{x}+\frac{1}{y}=\frac{1}{2} \Leftrightarrow \frac{x+y}{xy}=\frac{1}{2}$

$\Leftrightarrow xy = 2x + 2y \ (x; y \neq 0)$

$\Leftrightarrow x(y-2) = 2y \ (x; y \neq 0)$

$\Leftrightarrow x = \frac{2y}{(y-2)} \ (x; y \neq 0 \text{ or } 2)$

$\Leftrightarrow x = 2 + \frac{4}{y-2} \ (x; y \neq 0 \text{ or } 2)$

$\Leftrightarrow y-2 \in \{-4; -1; 1; 2; 4\}$, since x is an integer

All solutions: (1; –2); (–2; 1); (6; 3); (4; 4); (3; 6)

$y-2$	y	x
–4	–2	1
–1	1	–2
1	3	6
2	4	4
4	6	3

OR

$\frac{1}{x}+\frac{1}{y}=\frac{1}{2} \Leftrightarrow \frac{x+y}{xy}=\frac{1}{2}$

$\Leftrightarrow (x-2)(y-2) = 4$ with $x; y \neq 0$

All solutions: (1; –2); (–2; 1); (6; 3); (4; 4); (3; 6)

$x-2$	$y-2$	y	x
–1	–4	–2	1
–4	–1	1	–2
4	1	3	6
2	2	4	4
1	4	6	3
–2	–2	Invalid	

5. $B\hat{E}D = A\hat{B}E + x$ (external angle of Δ)

$\therefore A\hat{B}E = y - x$

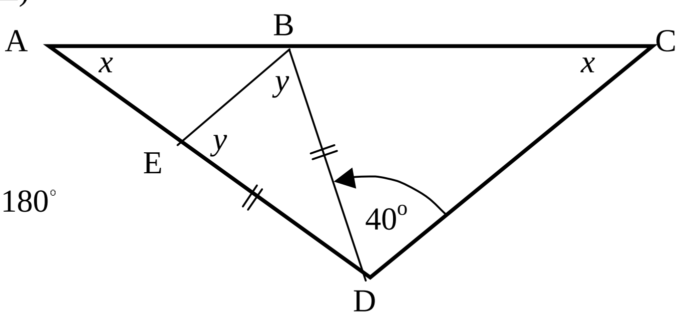
Now, $E\hat{D}B = 180^\circ - 2y$

So, $\hat{A} + \hat{C} + \hat{D} = x + x + (180^\circ - 2y) + 40^\circ = 180^\circ$

$\therefore 40^\circ + 2(x - y) = 0^\circ$

$\therefore y - x = 20^\circ$

i.e. $A\hat{B}E = 20^\circ$



6.

$$\frac{1}{2}+\left(\frac{1}{3}+\frac{2}{3}\right)+\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)+\left(\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}\right)+\cdots+\left(\frac{1}{100}+\frac{2}{100}+\frac{3}{100}+\cdots+\frac{99}{100}\right)$$

$$=\frac{1}{2}+1+\left(1+\frac{1}{2}\right)+2+\left(2+\frac{1}{2}\right)+\cdots+\left(49+\frac{1}{2}\right)$$

$$=\left(50\times\frac{1}{2}\right)+2\times[1+2+\cdots+49]$$

$$=25+2\times\frac{49\times50}{2}$$

$$=2475$$

OR

Note that $1+2+...+n=\frac{n(n+1)}{2}$

$$\therefore\frac{1}{2}+\left(\frac{1}{3}+\frac{2}{3}\right)+\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)+\left(\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}\right)+...+\left(\frac{1}{100}+\frac{2}{100}+\frac{3}{100}+...+\frac{99}{100}\right)$$

$$=\frac{1}{2}+\frac{2\times3}{2\times3}+\frac{3\times4}{2\times4}+...+\frac{99\times100}{2\times100}$$

$$=\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+...\frac{99}{2}$$

$$=\frac{99\times100}{2\times2}$$

$$=2475$$

7. $a=\sqrt{0,16}=(0,16)^{\frac{1}{2}}=0,4$

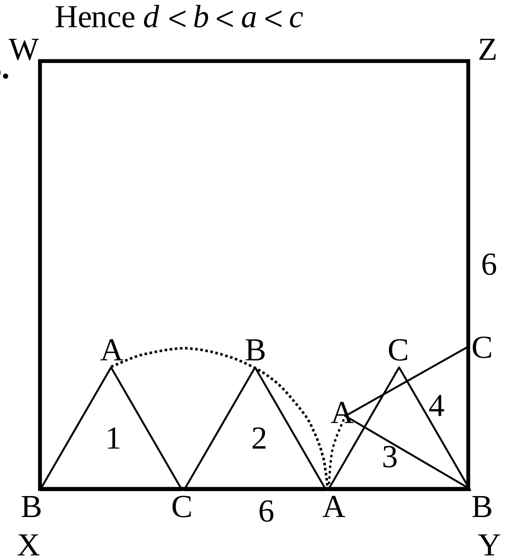
$b=\sqrt[3]{0,063}=(0,063)^{\frac{1}{3}}<(0,064)^{\frac{1}{3}}=0,4=a$

$c=\sqrt[5]{0,01025}=(0,01025)^{\frac{1}{5}}>(0,01024)^{\frac{1}{5}}=\left(\frac{1024}{10^5}\right)^{\frac{1}{5}}=\frac{2^2}{10}=0,4=a$

$d=(0,2)^2=[(0,2)^6]^{\frac{1}{3}}=(0,000064)^{\frac{1}{3}}<b$ OR $d^3=0,008<0,063$ $\therefore d<(0,063)^{\frac{1}{3}}=b$

Hence $d<b<a<c$

8.



The first four positions of the triangle are shown. During this part of the journey, A has travelled an arc length subtended by 120^{o} and another arc length subtended by 30^{o} (both radius 2). But A is now situated with respect to YZ exactly the same as it was relative to XY. So the total distance travelled by A is equivalent to $4\times120^{\circ}$ arc lengths plus $4\times30^{\circ}$ arc lengths (all radius 2), so an equivalent of $20\times30^{\circ}$ (or 600°) arc lengths or $\frac{5}{3}$ circles of radius 2.

Hence total length of path is $\frac{5}{3}\times4\pi=\frac{20\pi}{3}$.

9. $M \times A = 12$ (1)

$T \times H = 30$ (2)

$A \times H = 24$ (3)

$A \times T = 20$ (4)

$H \times S = 42$ (5)

From (2), (3) and (4): $(A \times H \times T)^2 = 30 \times 24 \times 20 = 14\,400$

$\Rightarrow A \times H \times T = 120$ (6)

Also, $\dfrac{(1)\times(5)}{(3)} = \dfrac{(M \times A)\times(H \times S)}{(A \times H)} = \dfrac{12 \times 42}{24} = 21$ (7)

$M \times A \times T \times H \times S = (A \times H \times T)\times(M \times S)$ (6)×(7)

$= 120 \times 21$

$= 2520$

OR

$M \times A \times T \times H \times S$

$= \dfrac{(M \times A)\times\sqrt{(A \times T)\times(T \times H} \times (H \times S)}{\sqrt{A \times H}}$

$= 12 \times \sqrt{\dfrac{20 \times 30}{24}} \times 42$

$= 12 \times \sqrt{25} \times 42$

$= 2520$

10. Every 2-digit number in this string must be one of 17, 34, 51, 68, 85 (multiples of 17) or 23, 46, 69, 92 (multiples of 23).

Suppose the string starts as 68517, then it cannot go any further.

So it must start as:

69234 69234 69234 …

i.e. cycles of length 5.

Since $2006 = 5 + 5 + 5 + \cdots + 5 + 1$, the 2006^{th} digit is the same as the first, so it is 6.

The number then is: 69234 6923469234 …69234692346,

therefore the last five digits are 92346

11. This can only be done by trial & error.

2	2	2	2	2
2	–9	2	–9	2
2	2	2	2	2
2	–9	2	–9	2
2	2	2	2	2

or

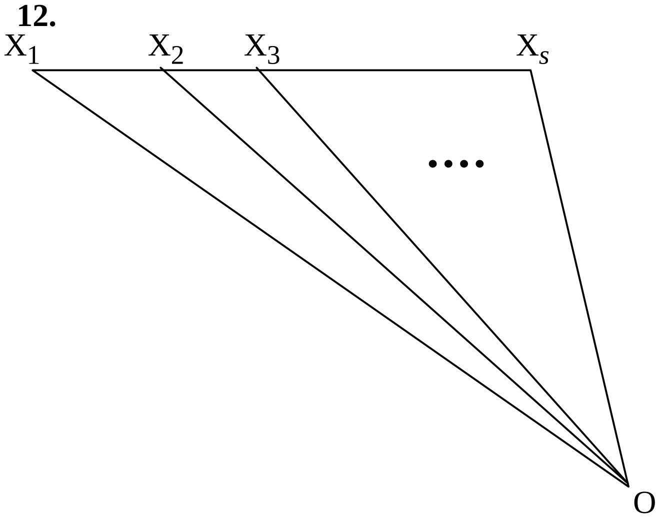
4	–1	2	–1	4
–1	–3	1	–3	–1
2	1	0	1	2
–1	–3	1	–3	–1
4	–1	2	–1	4

Every 2 by 2 block contains one –9 and three 2's, hence sum:
$-9 + 6 = -3 < 0$.
Since there are four –9's and 21 2's, the total sum is $21 \times 2 + 4(-9) = 6 > 0$.

Every 2 by 2 block has sum –1, but the total sum is 8.

The example and brief explanation are all that is required.

12.



Let's first consider the bottom horizontal line (as in the sketch).
There are $s - 1$ of the "small" triangles OX_1X_2; OX_2X_3; … ; $OX_{s-1}X_s$.
There are $s - 2$ of the triangles obtained by glueing two small ones together, i.e. OX_1X_3; OX_2X_4; … ; $OX_{s-2}X_s$.
Then $s - 3$ of those where three small ones are glued together, etc.
Finally, there is 1 triangle OX_1X_s where all small ones are glued together.

So the bottom horizontal line determines $(s-1) + (s-2) + \cdots + 2 + 1 = \frac{1}{2}s(s-1)$ triangles.

Each horizontal line determines this many triangles, so we have a total of $\frac{1}{2}hs(s-1)$ triangles.

13. The smallest possible value for *OAK* is 102 (we want it small to maximize the number of copies). $99\times102=10098$ (too big), while $98\times102, 97\times102, \cdots, 94\times102$ all contain multiple digits.

However, $93\times102=9486$ could be a solution.

Let's consider *OAK* = 103. Here we find

$97\times103=9991; 96\times103=9888; 95\times103=9785$ (a new max.) and $96\times105=10080$ (too large).

We also find that $104\times96=9\,984$ (with multiple digits), $104\times97=10\,088$ (too large), while $96\times m$, where $m\geq105$, is too large.

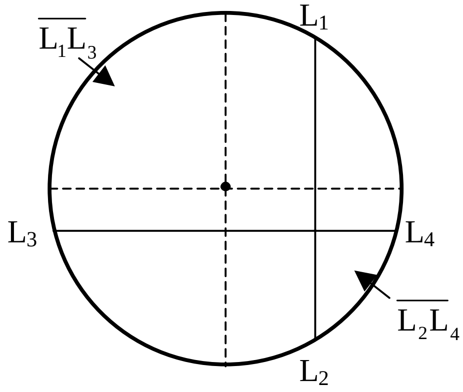
We conclude that when *OAK* = 103, we get the maximum number of copies giving:

$103+103+\cdots+103=9785=PINE$, i.e. 95 copies of *OAK*.

14. Let L_1L_2 and L_3L_4 be the cuts made by them. The dotted lines indicate the diameters parallel to these lines. Letters A to I indicate the areas of the regions in which they are written. Note that $B = E + H$ and $D = E + F$, so that $B \geq H$ and $D \geq F$. Now slice $\overline{L_1L_3}$ + slice $\overline{L_2L_4}$ has area

$A + B + D + E + I \geq A + H + E + F + I$

$= \frac{1}{2}$ area of pizza, since $B\geq H$ and $D \geq F$.



Andy should either choose slice

$\overline{L_1L_3}$ or slice $\overline{L_2L_4}$, (slice P or R in original pizza).

Andy should choose slices $\overline{L_1L_3}$ and $\overline{L_2L_4}$ (slices P and R in the original pizza)

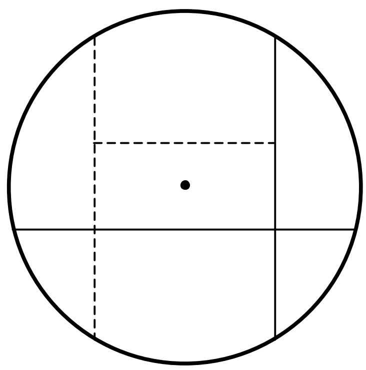
OR

Draw two extra lines symmetrically as shown

Then: $P_1 = S$, $P_2 = Q_2$ and $Q_1 = R$

$\therefore (P + R) - (Q + S) = (P_1 + P_2 + P_3 + R) - (Q_1 + Q_2 + S) = P_3 > 0$

Andy should choose slices P and R.



15. (a) 99999×66666

$=(10^6-1)\times 66\,666$

$=100\,000\times 66\,666-66\,666$

$=6\,666\,600\,000-66\,666$

$=6\,666\,533\,334$

$$\begin{array}{r} 6\,666\,600\,000 \\ -\ 66\,666 \\ \hline 6\,666\,533\,334 \end{array}$$

$\therefore$ Digit sum $=(6+3)\times4+(5+4)$

$=9\times4+9$

$=45$

(b) $999\cdots9\times kkk\cdots k$

$=(10^n-1)\times(k...k)$

$=kkk\cdots k000\cdots0$

$-\qquad kkk\cdots k$

$\underbrace{k...k}_{n-1}(k-1)\underbrace{(9-k)...(9-k)}_{n-1}(10-k)$

$\therefore$ Digit sum : $(n-1)k+(k-1)+(n-1)(9-k)+10-k$

$=(n-1)(k+9-k)+(k-1+10-k)$

$=9(n-1)+9$

$=9n$

(c) $\underline{m<n}$

$=(10^n-1)\times\dfrac{k}{9}(10^m-1)$

$=kkk\ ....k0..............00\ ....00$

$-\qquad\qquad k\ ......kk$

$\underbrace{k...k}_{m-1}(k-1)\underbrace{9...9}_{n-m}\underbrace{(9-k)...(9-k)}_{m-1}(10-k)$

$\therefore$ Digit sum : $(m-1)k+(k-1)+(n-m)9+(m-1)(9-k)+(10-k)$

$=(m-1)(k+9-k)+9(n-m)+(k-1+10-k)$

$=9(n-1+n-m+1)$

$=9n$

$\underline{m>n}$

$=(10^m-1)\times\dfrac{k}{9}(10^n-1)$

$=kkk\ ....k0..............00\ ....00$

$-\qquad\qquad k\ ......kk$

$\underbrace{k...k}_{n-1}(k-1)\underbrace{9...9}_{m-n}\underbrace{(9-k)...(9-k)}_{n-1}(10-k)$

$\therefore$ Digit sum : $(n-1)k+(k-1)+(m-n)9+(n-1)(9-k)+(10-k)$

$=(n-1)(k+9-k)+9(m-n)+(k-1+10-k)$

$=9(n-1+m-n+1)$

$=9m$

OR:

(b) $\underbrace{999\cdots9}_{n}\times\underbrace{kkk\cdots k}_{n}$

$=(10^n-1)\times(\underbrace{k...k}_{n})$

$=\underbrace{kkk\cdots k}_{n}\underbrace{000\cdots0}_{n}-\underbrace{kkk\cdots k}_{n}$

$\underbrace{k...k}_{n-1}(k-1)\underbrace{(9-k)...(9-k)}_{n-1}(10-k)$

where each of k, $k-1$, $9-k$, and $10-k$ represents a digit.

$\therefore$ Sum of digits $=(n-1)k+(k-1)+(n-1)(9-k)+10-k$

$=9n$

(c) The case of $m = n$ was treated in (b)

We may assume that $m < n$, since

$\underbrace{99...9}_{n}\times\underbrace{kk...k}_{m}=9\times\underbrace{11...1}_{n}\times k\times\underbrace{11...1}_{m}$

$=9\times\underbrace{11...1}_{m}\times k\times\underbrace{11...1}_{n}=\underbrace{99...9}_{m}\times\underbrace{kk...k}_{n}$

Now, $\underbrace{99...9}_{m}\times\underbrace{kk...k}_{n}$

$=(10^n-1)\times\underbrace{kk...k}_{n}$

$=\underbrace{kk...k}_{m}\underbrace{00...0}_{n}-\underbrace{kk...k}_{m}$

$=\underbrace{k...k}_{m-1}(k-1)\underbrace{9...9}_{n-m}\underbrace{(9-k)...(9-k)}_{m-1}(10-k)$

where each of k, $k-1$, $9-k$ and $10-k$ represents a digit

$\therefore$ Sum of digits $=(m-1)k+(k-1)+9(n-m)+(m-1)(9-k)+(10-k)$

$=9n$

We conclude that, in all cases ($m < n$, $m = n$, $m > n$), the sum of the digits is given by: $9\times\max\{m, n\}$.