SOUTH AFRICAN MATHEMATICS OLYMPIAD

Organised by the SOUTH AFRICAN MATHEMATICS FOUNDATION.

SECOND ROUND 2008
JUNIOR SECTION: GRADES 8 AND 9
ANSWERS AND SOLUTIONS

1.) B	2.) D	3.) D	4.) A	5.) D
6.) C	7.) A	8.) D	9.) B	10.) B
11.) E	12.) A	13.) B	14.) B	15.) B
16.) D	17.) D	18.) D	19.) E	20.) C

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SOLUTIONS

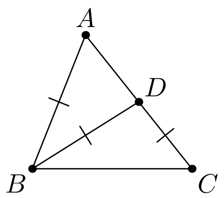
1. **Answer B.** Since $48 = 3 \times 16$ is a multiple of 3, we only need to check 1008 and 4008 (the only multiples of 3 in the list), of which only 1008 is divisible by 48.

2. **Answer D.** We have $16 = 16^1 = 4^2 = 2^4$, so the three pairs $(16, 1)$, $(4, 2)$ and $(2, 4)$ satisfy the equation $m^n = 16$.

3. **Answer D.**
$$3 - \frac{2}{3 - \frac{2}{3}} = 3 - \frac{2}{7/3} = 3 - \frac{6}{7} = \frac{21 - 6}{7} = \frac{15}{7}.$$

4. **Answer A.** The shares increase by $40\% \times$ R500 = R200 in value, so he sells them for R700. He pays $7\% \times$ R500 = R35 on the purchase price and $7\% \times$ R700 = R49 on the sell price. Thus his profit equals R700 - R500 - R35 - R49 = R116.

5. **Answer D.** Let $A\widehat{B}D = C\widehat{B}D = x$. Then $B\widehat{C}D = x$ $(BD = DC)$ and $A\widehat{D}B = 2x$ (sum of opposite interior angles). Since $AB = BD$, $B\widehat{A}D = B\widehat{D}A = 2x$. Now, in triangle ABD, the sum of the angles of the triangle is $2x + 2x + x = 5x = 180°$. So $B\widehat{A}D = 2x = 2 \times \frac{180°}{5} = 72°$.



6. **Answer C.** The three digit number equals $100(2x + 1) + 10(x - 1) + x = (200x + 100) + (10x - 10) + x = 211x + 90$.

7. **Answer A.** $23 = a^2 - b^2 = (a + b)(a - b) = 23(a - b)$, so $a - b = 1$, or $b = a - 1$. Thus $23 = a + b = a + (a - 1) = 2a - 1$, so $a = 12$.

8. **Answer D.** To obtain 10 zeroes at the end of $n!$, there must be 10 factors of $10 = 2\times 5$ in the product. Since there are more multiples of 2 than there are multiples of 5, we only have to look at the multiples of 5.

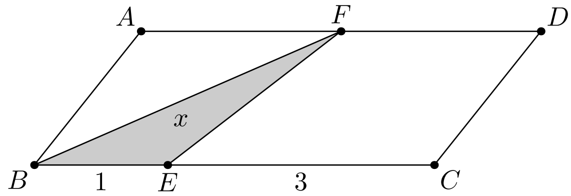
   Each multiple of 5 provides one factor 5, but 25 provides two factors 5, so we only need the first nine multiples of 5 in the product. Hence $n = 9 \times 5 = 45$.

9. **Answer B.** The area of the piece of perspex is $50 \times 32 = 1600 = 40^2$, so a square with the same area must have side length 40.

10. **Answer B.** Let the distance between Johannesburg and Cape Town be d. On the first journey, John takes $t_1 = \frac{d}{90}$ hours to complete the journey, and on the return journey, he takes $t_2 = \frac{d}{110}$ hours. Thus his total travel time is $t_1 + t_2 = \frac{d}{90} + \frac{d}{110} = \frac{110d+90d}{9900} = \frac{200d}{9900} = \frac{2d}{99}$. Thus his average speed is $\frac{\text{distance}}{\text{time}} = \frac{2d}{t_1+t_2} = \frac{2d}{2d/99} = 99$ km/h.

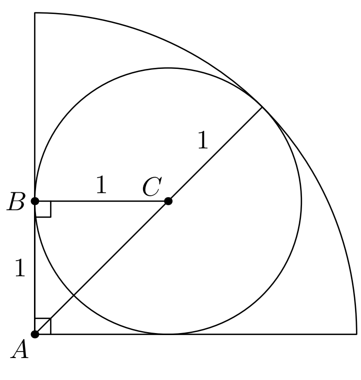
11. **Answer E.** We have $a < b$, $c < d$ and $b < d$. With this information we know that either a or c is the smallest, but it is impossible to determine which number is the smallest. For example, $a = 1,\ b = 2,\ c = 3,\ d = 4$ satisfies the given information (with a the smallest) and $a = 3, b = 4, c = 1, d = 5$ also satisfies the given information (with c the smallest).

12. **Answer A.** Let the area of the shaded triangle equal x. Because triangles BFE and BFC have the same height, we have that the area of triangle BFC is $4x$ (since its base is four times the base of triangle BFE). Now, the area of the parallelogram is equal to base $\times$ height, which is exactly twice the area of triangle BFC, so the area of parallelogram $ABCD = 2 \times 4x = 8x$. Thus $\frac{\text{area of } \triangle BEF}{\text{area of parallelogram } ABCD} = \frac{x}{8x} = \frac{1}{8}$.



13. **Answer B.** In any triangle, the sum of the two shortest sides is greater than the longest side. So the only sticks one can form a triangle with are the triples (2cm, 3cm, 4cm), (2cm, 4cm, 5cm) and (3cm, 4cm, 5cm).

14. **Answer B.** Use Pythagoras in triangle ABC: $AC^2 = AB^2 + BC^2 = 1 + 1 = 2$, so $AC = \sqrt{2}$. Then the radius of the big circle is equal to AC plus the radius of the small circle, so $R = 1 + \sqrt{2}$.



15. **Answer B.** In the first row there is 1 number.
In the first two there are $1 + 3 = 4 = 2^2$ numbers.
In the first three rows there are $1 + 3 + 5 = 9 = 3^2$ numbers.
So in the first 20 rows there are $20^2 = 400$ numbers. The last number in the 20^{th} row is the 400^{th} odd number, which is equal to $2(400) - 1 = 799$. By the same reasoning, the last number in the 19^{th} row is equal to $2(19^2) - 1 = 2(361) - 1 = 721$, so the first number in the 20^{th} row is 723. Now, the middle number in each row is equal to the

average of the first and the last number in that row. So the middle number in the 20^{th} row is equal to $\frac{723+799}{2} = 761$.

Alternatively, the sequence of middle numbers is:

$$1 = 0^2 + 1^2;\ 5 = 1^2 + 2^2;\ 13 = 2^2 + 3^2;\ 25 = 3^2 + 4^2;\ 49 = 4^2 + 5^2;\ \ldots$$

So the middle number in the 20^{th} row is equal to $19^2 + 20^2 = 361 + 400 = 761$.

16. **Answer D.** Divide the numbers from 1 to 2008 up into 11 groups according to what their remainders are when divided by 11. For example, the numbers 5, 16, 27, 38, ... are all in the same group because they all leave remainder 5 when divided by 11. Call the group of numbers which leave remainders of x when divided by 11 group x.

    Now, if a number from group x is in the new sequence, then no number in group $11 - x$ is allowed to be in the sequence, otherwise the sum of two numbers in those groups will leave remainder $x + 11 - x = 11$ when divided by 11, which means that it will be divisible by 11.

    Also, only one multiple of 11 may be included in the new sequence, because the sum of two multiples of 11 is divisible by 11.

    Also note that if one number of group $x \neq 0$ is in the new sequence, we can put them all into the new sequence. So the maximum number of numbers in the new sequence is obtained when we put in all the numbers in groups 1, 2, 3, 4 and 5 and one number from group 0.

    Now, $2008 = 11 \times 182$ remainder 6, so the first 6 groups all contain $182 + 1 = 183$ numbers. Thus the maximum number of numbers in the new sequence equals $5 \times 183 + 1 = 916$.

17. **Answer D.** Note that one line divides the plane into two parts, and four lines divide the plane into 11 parts. So we have the sequence

    $$2; \quad 4 = 2 + 2; \quad 7 = 4 + 3 = 2 + 2 + 3; \quad 11 = 7 + 4 = 2 + 2 + 3 + 4; \ldots.$$

    Thus n lines will divide the plane into $2+2+3+\cdots+n = 1+(1+2+3+\cdots+n) = 1+\frac{n(n+1)}{2}$ parts. So we solve the equation $1+\frac{n(n+1)}{2} = 172 \implies n(n+1) = 342 = 18\times 19$. So $n = 18$.

18. **Answer D.** The probability that the power will be not be cut on a given day, i.e. it will not be cut during peak hours nor during off-peak hours, is equal to $(1 - \frac{1}{7})(1 - \frac{1}{17}) = \frac{6}{7} \cdot \frac{16}{17} = \frac{6\times 16}{7\times 17} = \frac{96}{119}$.

    So the probability that the power will be cut on a given day equals $1 - \frac{96}{119} = \frac{119-96}{119} = \frac{23}{119}$.

19. **Answer E.** Suppose the wall contains x bricks. Thus Jeremy builds at a speed of $s_J = \frac{x}{16}$ bricks per hour, and Mpume builds at a speed of $s_M = \frac{x}{12}$ bricks per hour. Their combined speed, if they sometimes get into each other's way, is $s_J + s_M - 16$, but it is also equal to $\frac{x}{8}$. Hence $\frac{x}{16} + \frac{x}{12} - 16 = \frac{x}{8} \implies 16 = x(\frac{1}{16} + \frac{1}{12} - \frac{1}{8}) = x(\frac{3+4-6}{48}) \implies x = 16 \times 48 = 768$.

20. **Answer C.**

Let the box have dimensions x, y and z. Then $xyz = 2(xy+yz+zx) = 288 = 2^5 \times 3^2$. Thus x, y and z only has 2 and 3 as factors, and because there are three numbers and only two factors of 3, one of them, say x, is a power of 2.

Now, x divides xyz, so x must divide $2xy + 2yz + 2zx$ as well, which means x must divide $2yz$. Since there are only five factors of 2 that must be distributed between x, y and z and x must divide $2yz$, x can contain at most three factors of 2.

If $x = 2$, then $yz = 144$ and so $288 = 2xy + 2yz + 2zx = 4y + 288 + 4z$ which is impossible.

If $x = 8$, then $yz = 36$ and so $288 = 2xy + 2yz + 2zx = 16y + 72 + 16z$ which gives $y + z = 13\frac{1}{2}$, which is impossible if y and z are integers.

If $x = 4$, we have $yz = 72$ and $8y + 144 + 8z = 288$ which gives $y + z = 18$, and so $x + y + z = 22$. (Checking to make sure that y and z are integers, we calculate $y,\ z = 6, 12$.)