The Harmony South African Mathematics Olympiad
Junior First Round 2009: Solutions

1. **Answer D.**

2. **Answer C.** They sold 8 cakes more at the second social, which is $\frac{8}{50} = 16\%$ more than what they sold at the first social.

3. **Answer D.**
$$3 \div \frac{3}{8} = 3 \times \frac{8}{3} = 8.$$

4. **Answer B.** If we subtract the number of ears from the number of legs, we would have counted every animal exactly twice. Hence there are $\frac{92}{2} = 46$ animals.

   Alternative solution

   Suppose there are x animals. Then the number of legs they have is $4x$ and the number of ears they have is $2x$. So the difference between the two numbers is $4x - 2x = 2x = 92$. Hence there are $92 \div 2 = 46$ animals in the herd.

5. **Answer E.** Vusani's desk has 2 rows before it and 3 rows behind it, so there are $2 + 1 + 3 = 6$ rows of desks. Similarly there are $3 + 1 + 5 = 9$ left to right. So there are $6 \times 9 = 54$ desks in all.

6. **Answer A.** Pieter gets $\frac{7}{12}$ of the sweets, while Jacob gets $\frac{5}{12}$. Pieter thus has $\frac{7}{12} - \frac{5}{12} = \frac{2}{12} = \frac{1}{6}$ of the sweets more, which we know is equal to 14. Hence there are $6 \times 14 = 84$ sweets in the packet.

   Alternative solution

   If Pieter gets $7x$ sweets and Jacob gets $5x$ sweets, then $7x - 5x = 2x = 14$, so $x = 7$. The packet contained $(7 + 5)x = 12 \times 7 = 84$ sweets.

7. **Answer A.** The only possible pairs which could be reflections of each other are the pairs P and Q, and R and S. However, the diagonal parts of R and S slope in opposite directions, so they cannot be reflections of each other.

8. **Answer D.** Looking at the vertical lines in the figure, $CD + EF = AB = 20$, and similarly $BC + DE = AF = 25$. So the perimeter of the figure is $AB + BC + CD + DE + EF + FA = AB + (BC + DE) + (CD + EF) + FA = 20 + 25 + 20 + 25 = 90$.

9. **Answer D.** $PQ^2 = 2^2 + 1^2 = 5$, while $BC = 2 + 1 = 3$. So the ratio of the areas of the two squares equals $\frac{PQ^2}{BC^2} = \frac{5}{9}$.

10. **Answer C.** To compare the two given fractions, we write them with the same denominator. We have $\frac{4}{7} = \frac{32}{56}$ and $\frac{5}{8} = \frac{35}{56}$. Hence $\frac{33}{56}$ is the only fraction of the given ones lying between these two.

11. **Answer A.** If we add the numbers 21, 23 and 26, we would have counted all Liesl's toys exactly twice. So she has $\frac{21+23+26}{2} = \frac{70}{2} = 35$ toys, which means that she has $35 - 21 = 14$ jets.

    Alternative solution

    Suppose Liesl has j jets, t teddybears and c cars, and $j + t + c = x$ toys in total. Then

    $$j = x - 21, \quad t = x - 23, \quad c = x - 26.$$

    Adding these three equations yields $x = j + t + c = 3x - 70$. This simplifies to $2x = 70$ and so $x = 35$. Hence Liesl has $35 - 21 = 14$ jets.

12. **Answer D.** The 8-digit number is divisible by $18 = 2 \times 9$, so b must be even (for the number to be divisble by 2) and the sum of the digits must be divisible by 9 (for the number to be divisible by 9). The digit sum is equal to $27 + a + b$ which must be a multiple of 9, so $a + b$ must be a multiple of 9.

    Since a and b are nonzero and b is even, the only possibility is $a + b = 9$, and so the biggest possible difference is obtained when $a = 1$ and $b = 8$, giving a difference of $8 - 1 = 7$.

13. **Answer A.** By Pythagoras, the length of the third side of the triangle is equal to $\sqrt{10^2 - 6^2} = \sqrt{64} = 8$. Computing the area of the triangle in two ways gives the equation

    $$\tfrac{1}{2} \times 6 \times 8 = \tfrac{1}{2} \times 10 \times h,$$

    which simplifies to $h = \frac{24}{5}$.

14. **Answer C.** Once the position of the two red beads have been decided, all the other positions must contain green beads, so the arrangement is determined by the position of the red beads.

    If a red bead is in the first position, then there is 5 possible choices for the second red bead. If a red bead is in the second position, there are 4 possible positions for the second red bead, and so on. In total, there are thus $5 + 4 + 3 + 2 + 1 = 15$ possible arrangements.

    Alternative solution

    If we ignore the order of the two red beads on the wire, there is 6 possible positions for the one red bead, and 5 possible positions for the other, giving a total of $6 \times 5 = 30$ arrangements. However, swapping the two red beads doesn't change the arrangement, so we have to divide by 2. Thus there are $\frac{30}{2} = 15$ different arrangements.

15. **Answer D.** If the point P is back in its original position, it means that the point P has rolled around the small wheel a whole number of times, and the small wheel has rolled around the large wheel a whole number of times. The ratio of the small wheel's circumference to the large wheel's circumference is equal to $\frac{2}{5}$, so this means that the point P rolls around the small wheel 5 times, and the small wheel rolls around the large wheel twice. Hence the point P touches the circumference of the large wheel 5 times.

16. **Answer B.** Let $\angle D = x$. Then $\angle ECD = x$ ($EC = ED$) and so $\angle AEC = 2x$ (external angle of triangle ECD) and $\angle EAC = 2x$ ($AC = CE$). Triangle ABC is equilateral, so $\angle ACB = 60°$ and is also an external angle of triangle ACD. Hence $60° = \angle CAD + \angle CDA = x + 2x = 3x$ and so $x = 20°$.

17. **Answer D.** We calculate the area of the path by calculating the area of the straights and the corners separately. The four straight parts have area $2 \text{ m} \times 80 \text{ m} = 160 \text{ m}^2$ each.

    Since going around the track once means doing exactly a 360° turn, the four corners put together form a circle with radius two, which has area $\pi \times (2 \text{ m})^2 = 4\pi \text{ m}^2$. The total area of the track is thus $4 \times 160 \text{ m}^2 + 4\pi \text{ m}^2 = (640 + 4\pi) \text{ m}^2$.

18. **Answer D.** The number of numbers in each row is one more than in the previous row, so the total number of numbers in the triangle up to the n^{th} row is equal to $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

The number of numbers up the 62^{nd} row equals $\frac{62\times 63}{2} = 1953$. Up to the 63^{rd} row there are $1953+63 = 2016$ numbers, so the 2009^{th} number lies in row 63.

19. **Answer B.** We try looking for a pattern: $1^2 = 1$ with digit sum 1; $11^2 = 121$ with digit sum $4 = 2^2$; $111^2 = 12321$ with digit sum $9 = 3^2$. The pattern continues, so the digit sum of $(111\,111)^2$ is $6^2 = 36$.

    Note: the pattern only works for the first 9 terms. (Can you explain why?)

20. **Answer C.** We wish to find the positive values of n for which $\frac{3n-6}{n-1}$ is an integer. The fraction simplifies to $\frac{3n-6}{n-1} = \frac{3(n-1)-3}{n-1} = 3 - \frac{3}{n-1}$. This is an integer if and only if $n-1$ divides into 3. Also, since n is positive, $n-1$ is a positive divisor of 3, of which there are only two: 1 and 3, which corresponds to the two values $n = 2$ and $n = 4$.

    Alternative solution

    The given fraction can also be written as $\frac{3n-6}{n-1} = \frac{3(n-2)}{n-1}$, so $n-1$ must divide $3(n-2)$. Since $n-2$ and $n-1$ are consecutive integers, the greatest common divisor of $n-2$ and $n-1$ is equal to 1. This means that either $n-2 = 0$ (giving the solution $n = 2$) or $n-1$ must divide into 3, and the solution continues as in the first.